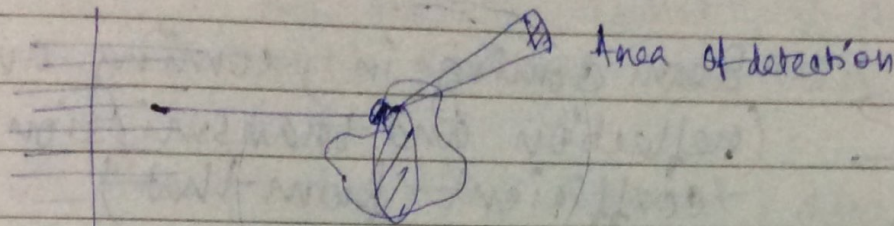


$\rightarrow$ spin $\frac{1}{2}$ (half integer)
 $p, n \rightarrow$ Baryons (heavier)
 $\pi^+, \pi^0, \pi^- \rightarrow$ Mesons (medium)
 $\rightarrow$ spin 1 (integer spin)

} Hadrons
 $\rightarrow$ takes part in strong interaction

If Baryon $\gamma = 1$, if not $\gamma = 0$ until the discovery of strange particles.

11/2/16



Linear cross section.

some of them hit this object and see how the beam particles scatter.

There is a cross section, in which the particles will hit, otherwise miss.

$$\frac{N_{\text{scattered}}}{N_{\text{incident}}} \times \text{Area of the beam} = \text{cross sectional area of the target}$$

$$\text{Flux}_{\text{inc}} = \frac{N_{\text{inc}}}{\text{Area}} / \text{time}$$

$$\rightarrow \frac{N_{\text{scattered}} / \text{time}}{\text{Flux}_{\text{inc}}}$$

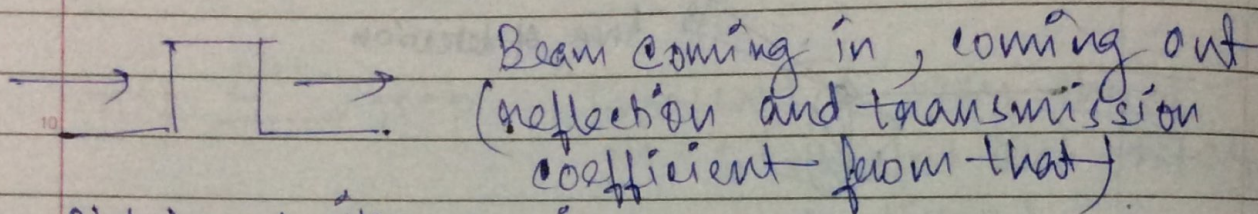
This is relevant if you have detector all around the object. But generally that is not the case.

Solid angle $= d\Omega$. (Detector is far away, and dividing by r^2).

Differential cross section $d\sigma = \left(\frac{d\sigma}{d\Omega} \right) d\Omega$

(fraction of particles which are going to be scattered within that particular solid angle)

Bigger the crosssection is, the more no. of particles are going to be scattered in that particular solid angle.



$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

$$\frac{\hbar^2 k^2}{2m} = E$$

$e^{ikx} \rightarrow$ coming from left

$e^{-ikx} \rightarrow$ coming from right

Incident beam, $\psi = (Ae^{ikx} + Be^{-ikx})e^{-i\omega t}$

↑ coming in. Reflected

On the other hand,

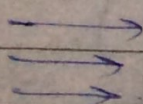
$$\psi = (Ce^{ikx} + D\cancel{e^{-ikx}})e^{-i\omega t}$$

↑ Transmitted wave

↑ This means that the beam is coming from right

Probability current $\rightarrow \vec{j} = \frac{\hbar}{m} \text{Im}(\psi^* \nabla \psi)$

(*) Let's consider, a spherically symmetric system (potential)



z axis

$$\psi \sim A e^{i\vec{k}\cdot\vec{r}} + f(\theta) \frac{e^{ikr}}{r}$$

↑ incident
↑ scattered (due to spherical symmetry)

If the potential had axial symmetry, then it would not have been simple.

Here, if we rotate about z axis, then, only ϕ would change and there is no ϕ dependence.

For, the probability current, due to the gradient term, we will have $\frac{1}{r^2}$ as well as $\frac{1}{r^3}$ term. It tells us, how many particles go into that cross-section $\rightarrow \left(\frac{A}{r^2}\right)$ just signifies the solid angle. But $\frac{1}{r^3}$ will be 0 (?) as you go far.

$$\vec{j}_{inc} = \frac{\hbar \vec{k}}{m} |A|^2$$

$$\vec{j}_{sc} = \frac{\hbar \vec{k}}{m} \frac{|f(\theta)|^2}{r^2} \hat{n} |A|^2$$

If you multiply this by area of the detection, r^2 will get cancelled

$$d\sigma = \frac{j_{sc}}{j_{inc}} \times \text{Area} = \frac{|f(\theta)|^2}{r^2} \cdot A = |f(\theta)|^2 d\Omega$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

* Optical theorem

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{4\pi}{k} \text{Im}(f(0))$$

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) f_l P_l(\cos\theta)$$

The incident wave can be expanded,

$$e^{ikz} = e^{ikr \cos\theta} = \sum_{l=0}^{\infty} (2l+1) j_l(kr) i^l P_l(\cos\theta)$$

Spherical symmetry $\Rightarrow$ angular momentum conservation

$$j_l(kr) \sim (kr)^l \quad (kr \gg 0)$$

$$\sigma = \int_{-1}^{+1} |f(\theta)|^2 d\cos\theta$$

$$= 2\pi \int_{-1}^{+1} d(\cos\theta) \sum_{l,l'} (2l+1)(2l'+1) P_l(\cos\theta) P_{l'}(\cos\theta) f_l f_{l'}^*$$

$$= 4\pi \sum_l (2l+1) |f_l|^2$$

$$\text{Im}(f(\theta)) = \sum_l (2l+1) \text{Im}(f_l)$$

$$P_l(1) = 1$$

From optical theorem, $|f_l|^2 = \frac{1}{k} \text{Im}(f_l)$

$$|kf_l|^2 = \text{Im}(kf_l)$$

$$z = x + iy; |z|^2 = x^2 + y^2 = y \quad \text{for } |z|^2 = \text{Im}(z)$$

$$\Rightarrow x^2 + (y - \frac{1}{2})^2 = \frac{1}{4} \quad \left(\text{A circle, centered at } (0, \frac{1}{2}) \text{ having radius } \frac{1}{2} \right)$$

$$|z - z_0| = R$$

$$|z - \frac{1}{2}| = \frac{1}{2}$$

$$|2z - i| = 1 \Rightarrow |1 + 2iz| = 1$$

$$\Rightarrow |1 + 2iKf_L| = 1$$

$$1 + 2iKf_L = e^{i2\delta_L}$$

$$\text{Then, } f_L = \frac{e^{i2\delta_L} - 1}{2iK} = \frac{e^{i\delta_L} \sin \delta_L}{K}$$

$$\frac{1}{f_0} = \frac{K e^{-i\delta_0}}{\sin \delta_0} \quad \text{Im}\left(\frac{1}{f_0}\right) = -K \Rightarrow \frac{1}{f_0} = (\eta - i)K$$

$$\text{If there is a peak (resonance) } \eta = \frac{E - E_0}{\text{const}} (?)$$

$$\text{Im}\left(\frac{1}{Kf_L}\right) = -1$$

$$\frac{1}{Kf_L} = (\eta - i) \quad f_0 = \frac{1}{(\eta - i)K}$$

At resonance, $\eta = 0$.

Near resonance, $\eta \propto (E - E_0)$.

$$24/2/16 \quad \text{Im}\left(\frac{1}{f}\right) = -iK$$

$$\frac{1}{f} = (\eta - i)K \quad \eta \in \mathbb{R}$$

$$\sigma = |f|^2$$

Max. scattering occurs at $E = E_0$ for which $\eta = 0$

Near E_0 ,

$$\eta = \frac{1}{T} (E_0 - E)$$

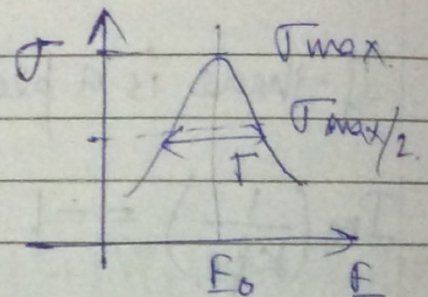
$$f = \frac{1}{K} \frac{1}{(E_0 - E) \frac{\Gamma}{2} - i}$$

$$= \frac{1}{K} \frac{\Gamma/2}{(E_0 - E) - i\Gamma/2}$$

Breit-Wigner form

$$\sigma = \frac{1}{K^2} \frac{(\Gamma/2)^2}{(E - E_0)^2 + (\Gamma/2)^2}$$

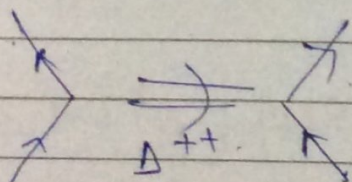
$$\sigma = \sigma_{\max} \frac{(\Gamma/2)^2}{(E - E_0)^2 + (\Gamma/2)^2}$$



At $E - E_0 = \pm \Gamma/2$ we get the half value of the maximum cross section.

$$\sigma = \frac{\sigma_{\max}}{2} \quad \Gamma \rightarrow \text{full width at half max.}$$

We need a relativistic generalization of this.
Decay can't be unitary



proton-proton scattering

$$\frac{1}{p_A^2 - m_A^2} \leftarrow \text{Propagation for free particle}$$

If there is interaction, $\frac{1}{p_A^2 - m_A^2 + i\Gamma}$ $\sim \frac{1}{p_A^2 - m_A^2 - i\Gamma_{\max}}$

For $p^2 = m^2$ there is a jump. (?)
 So, if we see a peak at one particular energy, then we can say that there is a particle which decayed at that energy.
 (peak indicates mass)

Spread in the energy of the decaying particle
 $\rightarrow$ Lifetime of the particle (width indicates lifetime)



$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu$$

$$(\Lambda, a)(\bar{\Lambda}, \bar{a}) = (\Lambda\bar{\Lambda}, \Lambda\bar{a} + a)$$

If $\Lambda = \mathbb{I}$, then,

$$(\mathbb{I}, a)(\mathbb{I}, \bar{a}) = (\mathbb{I}, a + \bar{a}) \rightarrow \text{Translations}$$

$$(\Lambda, 0)(\bar{\Lambda}, 0) = (\Lambda\bar{\Lambda}, 0) \rightarrow \text{Lorentz transformation}$$

Semi-direct product

$$|\psi\rangle \rightarrow |\psi'\rangle = U(\Lambda, a)|\psi\rangle$$

This is unitary (?) [Weinberg]

$$\boxed{\Lambda^T \eta \Lambda = \eta} \rightarrow \det \Lambda = \pm 1$$

$$|\Lambda^0_0| \geq 1$$

Proper $\rightarrow \det \Rightarrow 1 \rightarrow$ connected to identity

$$\boxed{\det \Lambda = +1}$$

$$\Lambda^0_0 \geq +1$$

Proper Lorentz group

$$U(\Lambda, a) U(\bar{\Lambda}, \bar{a}) = U(\Lambda \bar{\Lambda}, \Lambda \bar{a} + a) \leftarrow \text{modulo phase factors}$$

Proper infinitesimal Poincare transformation

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$$

We have $\omega_{\mu\nu} + \omega_{\nu\mu} = 0$ (antisymmetric)
 $\hookrightarrow$ six independent parameters

$$\omega^\mu{}_\nu = \epsilon^\mu{}_\nu \quad (4 \text{ parameters for 4 coordinates})$$

$$U(\mathbb{I} + \Omega, \epsilon) = \mathbb{I} + \frac{i}{2} \omega_{\mu\nu} J^{\mu\nu} - i \epsilon_\mu P^\mu + \dots$$

$$J^{\mu\nu} = -J^{\nu\mu}$$

$$(J^{\mu\nu})^\dagger = J^{\mu\nu}, \quad (P^\mu)^\dagger = P^\mu$$

The quantities with parameters Λ and a ,

$$(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a)$$

$$U(\Lambda, a) U(\bar{\Lambda}, \bar{a}) U^{-1}(\Lambda, a)$$

$$= U(\Lambda \bar{\Lambda}, \Lambda \bar{a} + a) U(\Lambda^{-1}, -\Lambda^{-1}a)$$

$$= U(\Lambda \bar{\Lambda} \Lambda^{-1}, \Lambda \bar{a} + a - \Lambda \bar{\Lambda} \Lambda^{-1}a)$$

$$\text{For } \bar{\Lambda} = \mathbb{I} + \Omega \quad \text{and } \bar{a} = \epsilon.$$

$$U(\Lambda, a) U(\mathbb{I} + \Omega, \epsilon) U^{-1}(\Lambda, a)$$

$$= U(\Lambda \bar{\Lambda}, \Lambda \bar{a} +$$

$$= U(\mathbb{I} + \Lambda \Omega \Lambda^{-1}, \Lambda \epsilon - (\Lambda \Omega \Lambda^{-1}) a)$$

This is an infinitesimal transformation. Then,

$$= \mathbb{I} + \frac{i}{2} (\Lambda \Omega \Lambda^{-1})_{\mu\nu} J^{\mu\nu} - i (\Lambda \epsilon - \Lambda \Omega \Lambda^{-1} a)_m P^m$$

$$U(\Lambda, a) \left(\mathbb{I} + \frac{i}{2} \omega_{\rho\sigma} J^{\rho\sigma} - i \epsilon_\rho P^\rho \right) U^{-1}(\Lambda, a)$$

$$= \mathbb{I} + \frac{i}{2} (\Lambda \Omega \Lambda^{-1})_{\mu\nu} J^{\mu\nu} - i (\Lambda \epsilon - \Lambda \Omega \Lambda^{-1} a)_m P^m$$

$$U(\Lambda, a) J^{\rho\sigma} U^{-1}(\Lambda, a) = \Lambda_\mu^\rho \Lambda_\nu^\sigma \left(J^{\mu\nu} - a^\mu p^\nu + a^\nu p^\mu \right)$$

$$U(\Lambda, a) P^\rho U^{-1}(\Lambda, a) = \Lambda_\mu^\rho P^\mu$$

$$\textcircled{*} \frac{1}{2} \omega_{\mu\nu} U(\Lambda, a) J^{\mu\nu} U^{-1}(\Lambda, a)$$

$$= \frac{i}{2} \omega_{\rho\sigma} (\Lambda_\mu^\rho \Lambda_\nu^\sigma J^{\mu\nu}) - \omega_{\rho\sigma} \Lambda_\mu^\rho \Lambda_\nu^\sigma a^\mu p^\nu$$

$$i [J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho} J^{\mu\sigma} + \eta^{\mu\sigma} J^{\nu\rho} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho}$$

$$i [P^\mu, J^{\rho\sigma}] = \eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho$$

$$[P^\mu, P^\nu] = 0$$

$$\vec{J} = \{J^{23}, J^{31}, J^{12}\}$$

$$\vec{K} = \{J^{01}, J^{02}, J^{03}\}$$

$$\vec{P} = \{P^1, P^2, P^3\}$$

$$H = P^0$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k$$

$$[K_i, P_j] = iH\delta_{ij}$$

$$[J_i, P_j] = i\epsilon_{ijk} P_k$$

$$P^\mu |p, \sigma\rangle = p^\mu |p, \sigma\rangle$$

↑
4 momentum
operator

↑
its
eigenstate

↑
eigenvalue

$$U(\mathbb{T}, a) = e^{-iP \cdot a} \quad \left| \quad \lim_{N \rightarrow \infty} \left(\mathbb{I} - \frac{1}{N} a_\mu P^\mu \right)^N \right.$$

$$= e^{-i a_\mu p^\mu}$$

operator for finite translation

$$e^{-iP \cdot a} |p, \sigma\rangle$$

$$= e^{-i p^\mu a_\mu} |p, \sigma\rangle$$

25/2/16 Ref. Weinberg. QFT vol. 1

$$U(\Lambda) = U(\Lambda, 0)$$

$$U^{-1}(\Lambda) P^\mu U(\Lambda) = \Lambda^\mu_\nu P^\nu \quad [P_\mu, P_\nu] = 0$$

$$U(\Lambda, a) = U(\mathbb{T}, a) U(\Lambda, 0)$$

$$P^\mu |p, \sigma\rangle = p^\mu |p, \sigma\rangle$$

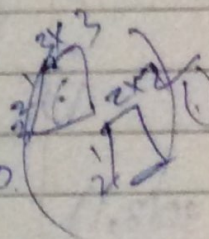
$$U(\mathbb{T}, a) |p, \sigma\rangle = e^{-i a \cdot p} |p, \sigma\rangle$$

$$U(N) |p, \sigma\rangle = ?$$

$$\begin{aligned} P^N (U(N) |p, \sigma\rangle) &= U(N) [U^{-1}(N) P^N U(N)] |p, \sigma\rangle \\ &= U(N) \Lambda^N \rho P^L |p, \sigma\rangle \\ &= \underbrace{(\Lambda^N \rho P^L)}_{(\Lambda P)^N} [U(N) |p, \sigma\rangle] \end{aligned}$$

$$U(N) |p, \sigma\rangle = \sum_{\sigma'} C_{\sigma'\sigma}(N, p) |\Lambda p, \sigma'\rangle$$

$C_{\sigma'\sigma}(N, p) |\Lambda p, \sigma'\rangle$ can be reduced to block diagonal form.



$$\boxed{|p, 1\rangle} = \alpha |\Lambda p, 1\rangle + \beta |\Lambda p, 2\rangle + \gamma |\Lambda p, 3\rangle$$

↑
Loorentz transf. We are not considering $|\Lambda p, 4\rangle$ and $|\Lambda p, 5\rangle$

In the block diagonal form, the 4th and 5th elements are not affected by the transf.

So, $|\Lambda p, 4\rangle, |\Lambda p, 5\rangle$ are states of a different particle.

Wigner's method of induced representation

Proper Loorentz transformation doesn't change the sign of p^0 if it is time-like.

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ M \end{pmatrix} \rightarrow \text{Invariant under rotation}$$

$M \leftarrow 0^{\text{th}} \text{ component}$

Lorentz transformation doesn't change the norm

SO(3) $\begin{pmatrix} 0 \\ 0 \\ 0 \\ M \end{pmatrix} \rightarrow$ gives all states norm M^2 . (M^2) 0
& p^0 has positive sign

ISO(2) $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \rightarrow p^0$ All lightlike momenta can be obtained through this. (photons)

SO(3,1) $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow$ vacuum

$$p^\mu = [L(p)]^\mu_\nu k^\nu$$

$\uparrow$ standard

Many Lorentz transformations give the same p^μ starting from k^ν .

$$p^2 = k^2, \text{sgn}(p^0) = \text{sgn}(k^0)$$

$$U(L(p)) |k, \sigma\rangle \equiv |p, \sigma\rangle$$

$\uparrow$ specific choice of $L(p)$. (not all $L(p)$ s)

$$U(\Lambda) |p, \sigma\rangle = U(\Lambda L(p)) |k, \sigma\rangle$$

$$= U(L(\Lambda p)) \underbrace{U(L^{-1}(\Lambda p) \Lambda L(p))}_{W(\Lambda, p)} |k, \sigma\rangle$$

Any rotation if we start from k . for $\begin{pmatrix} 0 \\ 0 \\ 0 \\ M \end{pmatrix}$

Little group of $k \equiv \{ \text{All L.T.s that leave } k \text{ the same} \}$

$$U(\Lambda) |p, \sigma\rangle = U(L(\Lambda p)) U(W(\Lambda, p)) |k, \sigma\rangle$$

$$U(W(\Lambda, p)) |k, \sigma\rangle$$

$$\langle k, \sigma | k', \sigma' \rangle = \delta_{\sigma\sigma'} \delta^3(\vec{k} - \vec{k}')$$

↓
This norm is not Lorentz invariant
So, we don't want $N(p)$ to be 1

$$U(W(\Lambda, p)) |k, \sigma\rangle = \sum_{\sigma'} D_{\sigma\sigma'}(W(\Lambda, p)) |k, \sigma'\rangle$$

$$U(\Lambda) |p, \sigma\rangle = N(p) U(L(\Lambda p)) \sum_{\sigma'} D_{\sigma\sigma'}(W(\Lambda, p)) |k, \sigma'\rangle$$

$$= \frac{N(p)}{N(\Lambda p)} \sum_{\sigma'} D_{\sigma\sigma'}(W(\Lambda, p)) |k, \sigma'\rangle$$

↓
 $D_{\sigma\sigma'}$ of $W(\Lambda, p)$

29/2/16 $U(\Lambda, a) |p, \sigma\rangle$

$$U(\Lambda, a) = U(I, a) U(\Lambda)$$

We want to find out the state $|k, \sigma\rangle$

We start with standard 4-momentum k^μ
 Fix: $L(p) \leftarrow$ A Lorentz transformation

$$p^\mu = L(p)^\mu{}_\nu k^\nu$$

Define, $|p, \sigma\rangle = N(p) U(L(p)) |k, \sigma\rangle$ (The σ label has to be same on both sides)
 Normalization

$$\begin{aligned} U(\Lambda) |p, \sigma\rangle &= N(p) U(\Lambda L(p)) |k, \sigma\rangle \\ &= N(p) U(L(\Lambda p)) U(\underbrace{L^{-1}(\Lambda p) \Lambda L(p)}_{W(\Lambda, p)}) |k, \sigma\rangle \end{aligned}$$

A Lorentz transformation which acts on k and gives you back k .
 called little group of k

$$= \frac{N(p)}{N(\Lambda p)} \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, p)) |\Lambda p, \sigma'\rangle$$

$$U(W(\Lambda, p)) |k, \sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, p)) |k, \sigma'\rangle$$

$$D_{\sigma'\sigma}(\Lambda, p) = \frac{N(p)}{N(\Lambda p)} D_{\sigma'\sigma}(W(\Lambda, p))$$

$$N(p) = \sqrt{\frac{k^0}{p^0}} \leftarrow \text{for a particular choice.}$$

Massive Particles

Norm = M^2 , simplest possibility of 4 momentum. $k = \begin{pmatrix} 0 \\ 0 \\ 0 \\ M \end{pmatrix}$

As the 0th component is not changed $\Rightarrow$ rotation.
 $W = SO(3)$

If 2 states $\rightarrow$ not connected by any Lorentz transformation $\Rightarrow$ then you can think that they are 2 different particles.

4x4 $\rightarrow$ matrix $L(p)$ acting on k , should give $p = \begin{pmatrix} \vec{p} \\ m \end{pmatrix}$

$$L(p) = \begin{pmatrix} \hat{p} \beta \gamma \\ \gamma \end{pmatrix}$$

$$|\vec{p}| = M \beta \gamma$$

$$\frac{\vec{p}}{M} = \frac{|\vec{p}| \hat{p}}{M}$$

Here, $[L(p)]^0_0 = \gamma$ and $[L(p)]^i_0 = \hat{p}_i \beta \gamma$

For fixing the other 3 columns, only the condition of Lorentz transformation is not enough. How to find them uniquely?

Let's consider a particle moving along Z axis.

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ M \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ \beta \gamma M \\ \gamma M \end{pmatrix}$$

A standard L.T for this, $B(|\vec{p}|) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \beta \gamma \\ 0 & 0 & \beta \gamma & \gamma \end{pmatrix}$

$R(\hat{p})$ takes $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 0 \\ \hat{p} \\ 1 \end{pmatrix}$ (arbitrary direction of $\vec{p}$)

$$L(\hat{p}) = R(\hat{p}) B(|\hat{p}|) R^{-1}(\hat{p})$$

Rotate to $\hat{z}$, boost, Rotate -1

doesn't change the time component

$$L(\hat{p}) = \left(\begin{array}{c|c} R_3(\hat{p}) & 0 \\ \hline 0 & 1 \end{array} \right) \left(\begin{array}{c|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma \\ \hline \gamma \hat{p} \hat{e}_3^T & \gamma \end{array} \right) \left(\begin{array}{c|c} R_3^T(\hat{p}) & 0 \\ \hline 0 & 1 \end{array} \right)$$

$$R_3(\hat{p}) \hat{e}_3 = \hat{p}$$

$R(\hat{p})$ acting on $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ will give a vector $\perp$ to $\hat{p}$, as $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ were $\perp$ at the first place

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma \end{pmatrix} = I_3 + (\gamma - 1) \hat{e}_3 \hat{e}_3^T \quad \left| \quad \hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right.$$

Substituting this in $L(\hat{p})$, and multiplying,

$$L(\hat{p}) = \left(\begin{array}{c|c} I_3 + (\gamma - 1) \hat{p} \hat{p}^T & \gamma \hat{p} \hat{e}_3^T \\ \hline \gamma \hat{p} \hat{p}^T & \gamma \end{array} \right)$$

So, in our case, $L(\hat{p})^0_0 = \gamma$, $L(\hat{p})^i_0 = L(\hat{p})^0_i = \hat{p}_i \gamma$
 $L(\hat{p})^i_j = \delta_{ij} + (\gamma - 1) \hat{p}_i \hat{p}_j$

A notation is a Lorentz Transformation

$$\Lambda = R \Rightarrow W(R, \hat{p}) = L^{-1}(R\hat{p}) R L(\hat{p})$$

$$= [R(R\hat{p}) B(|R\hat{p}|) R^{-1}(R\hat{p})]^{-1}$$

$$R R(\hat{p}) B(|\hat{p}|) R^{-1}(\hat{p})$$

Rotation doesn't
change magnitude

$$= R(R\hat{p})B^{-1}(|\hat{p}|)R^{-1}(R\hat{p}) \\ R(R\hat{p})B(|\hat{p}|)R^{-1}(\hat{p})$$

$$R^{-1}(R\hat{p})R(R\hat{p})$$

↳ Takes $\hat{p}$ to $\hat{p}$

↳ Takes $\hat{p}$ to $\hat{p}$

belongs to little group of rotation. R

Then, $B^{-1}(|\hat{p}|)B(|\hat{p}|)$ would be cancelled. choosing $R(\hat{p})$ gives us the same maths for relativistic case as in non-relativistic case.

2/3/16 $M=0$

standard K , and $\hat{p} = L(\hat{p})K$

$$N(\hat{p})U(L(\hat{p}))|K, \sigma\rangle = |\hat{p}, \sigma\rangle$$

$$U(\Lambda)_{|\hat{p}, \sigma\rangle} = \frac{N(\hat{p})}{N(\Lambda\hat{p})} \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, \hat{p})) |\Lambda\hat{p}, \sigma'\rangle$$

$$W(\Lambda\hat{p}) = L^{-1}(\Lambda\hat{p})\Lambda L(\hat{p})$$

$$M=0 \rightarrow 0 \text{ Norm}$$

$$K = \begin{pmatrix} 0 \\ 0 \\ K \\ K \end{pmatrix}$$

And sign of K^0 will change remain unchanged.

$$\text{Vacuum, } -K = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \text{Trivial case}$$

$$U(\Lambda)|\text{vac}\rangle \rightarrow |\text{vac}\rangle$$

Find a Lorentz transformation which acting on $\begin{pmatrix} 0 \\ 0 \\ K \\ K \end{pmatrix}$ gives you the same thing back.

$$WK = K \quad K = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Find out what W is. (We can't choose $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ as it doesn't have the dimension of K , for getting $\uparrow$)
But here we are taking it for convenience.

$$t = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \leftarrow \text{Time like vector.}$$

$$(Wt)^{\mu} (Wt)_{\mu} = t^{\mu} t_{\mu} = 1 \quad \left(\begin{array}{l} W \text{ is a Lorentz transformation, it keeps the inner product invariant.} \end{array} \right)$$

$$(Wt)^{\mu} K_{\mu} = (Wt)^{\mu} (WK)_{\mu} = t^{\mu} K_{\mu} = 1 \quad [WK = K]$$

$$Wt = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ 1+\gamma \end{pmatrix} \quad \text{It, when inner product is taken with } \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \text{ gives } 1.$$

α, β are arbitrary parameters.
 γ is not. $\rightarrow \gamma = \frac{\alpha^2 + \beta^2}{2}$ [from the constraint $(Wt)^{\mu} (Wt)_{\mu} = 1$]

$$\begin{pmatrix} -\alpha & \alpha \\ -\beta & \beta \\ 1-\gamma & \gamma \\ -\gamma & 1+\gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

For getting $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ on the right side.

We don't get information about the first 2 columns from this. But they should be orthogonal to the 3rd, 4th column, and should

have norm 1 (for Lorentz transformation)
 But there are infinitely many possibilities.
 If 2 orthogonal vectors are there, their
 linear combination is also orthogonal.

Choosing, $\begin{pmatrix} 1 \\ 0 \\ x \\ y \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ p \\ q \end{pmatrix}$ as the first 2

columns and then checking the orthogonality
 condition we get.

$$S(\alpha, \beta) = \begin{pmatrix} 1 & 0 & -\alpha & \alpha \\ 0 & 1 & -\beta & \beta \\ \alpha & \beta & 1-\gamma & \gamma \\ \alpha & \beta & -\gamma & 1+\gamma \end{pmatrix}$$

$$S^{-1}(\alpha, \beta)W : \begin{matrix} t \rightarrow t \\ k \rightarrow k \end{matrix} \quad \left(S(\alpha, \beta) \text{ does the same work as } W \right)$$

Leaves k the same $\Rightarrow$ rotation

And leaves $\begin{pmatrix} 0 \\ 0 \\ k \\ k \end{pmatrix}$ unchanged $\Rightarrow$ rotation about z axis.

$$S^{-1}(\alpha, \beta)W = R(\theta)$$

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is the structure of the Little group.

$$W(\alpha, \beta, \theta) = S(\alpha, \beta) R(\theta)$$

We know, $R(\theta) R(\theta') = R(\theta + \theta')$

And, $S(\alpha, \beta) S(\alpha', \beta') = S(\alpha + \alpha', \beta + \beta')$

↑
special kind of element
in the little group.

Take all possible $S(\alpha, \beta) \Rightarrow$ forms a subgroup of the whole little group. $\rightarrow$ 2 parameters.
group $\Rightarrow$ translation in 2 dimension.
 $\Rightarrow$ isomorphic to translation.

$R(\theta)$ is not isomorphic to translation. There is bounds on the parameter θ .

$$R(\theta) S(\alpha, \beta) R^{-1}(\theta) = S(\alpha \cos \theta + \beta \sin \theta, -\alpha \sin \theta + \beta \cos \theta)$$

Set of all $S(\alpha, \beta)$ forms a normal subgroup.

$$S(\alpha', \beta') R(\theta) S(\alpha, \beta) R^{-1}(\theta) S(\alpha', \beta') \\ = S(\alpha \cos \theta + \beta \sin \theta, -\alpha \sin \theta + \beta \cos \theta)$$

$S(\alpha', \beta')$ $\rightarrow$ arbitrary element doesn't change this expression, as the elements of S subgroup commutes with each other.

$\Rightarrow$ Abelian invariant, not semi-simple

ISO(2, 1).

Infinitesimal little group elements

if $\alpha, \beta, \theta \neq 0$, then $w(\alpha, \beta, \theta) = \mathbb{I}$

$$w(\alpha, \beta, \theta) = \begin{pmatrix} 1 & 0 & -\alpha & \alpha \\ 0 & 1 & -\beta & \beta \\ \alpha & \beta & 1 & 0 \\ \alpha & \beta & 0 & 1 \end{pmatrix}$$

↑

comes from

$$S(\alpha, \beta) R(\theta) = \begin{pmatrix} 1 & 0 & -\alpha & \alpha \\ 0 & 1 & -\beta & \beta \\ \alpha & \beta & 1 & 0 \\ \alpha & \beta & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\left[\hat{S} \text{ is second order } \left(\frac{\alpha^2 + \beta^2}{2} \right) \right]$$

$w(\alpha, \beta, \theta)$ is a Lorentz transformation.

$$A^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$$

$$\Rightarrow U = \mathbb{I} + \frac{i}{2} \omega_{\mu\nu} J^{\mu\nu}$$

$$U(w(\alpha, \beta, \theta))$$

$$= \mathbb{I} + i\alpha A + i\beta B + i\theta J_3$$

$$\text{where, } A = -J^{13} + J^{10} = J_2 + K_1$$

$$B = -J^{12} + J^{10} = -J_1 + K_2$$

$$A, B, J_3 \Rightarrow [J_3, A] = iB \quad [A, B] = 0$$

$$[J_3, B] = -iA$$

$$U(R(\theta)) U(S(\alpha, \beta)) U^{-1}(R(\theta)) \\ = U(S(\alpha \cos \theta + \beta \sin \theta, -\alpha \sin \theta + \beta \cos \theta))$$

Let's say α, β are infinitesimal
Then,

$$U(R(\theta)) (\mathbb{I} + i\alpha A + i\beta B) U^{-1}(R(\theta)) \\ = \mathbb{I} + i(\alpha \cos \theta + \beta \sin \theta) A + i(-\alpha \sin \theta + \beta \cos \theta) B$$

$$U(R(\theta)) A U^{-1}(R(\theta)) = A \cos \theta - B \sin \theta$$

$$U(R(\theta)) B U^{-1}(R(\theta)) = A \sin \theta + B \cos \theta$$

These results can also be derived from
Baker-Hausdorff formula.

$$A |K, a, b, \dots\rangle = a |K, a, b, \dots\rangle \quad \text{common eigenstates as } [A, B] = 0$$

$$B |K, a, b, \dots\rangle = b |K, a, b, \dots\rangle \quad \{a, b \text{ are labels}\}$$

$$A U^{-1}(R(\theta)) |K, a, b, \dots\rangle = U^{-1}(R(\theta)) (A \cos \theta - B \sin \theta) |K, a, b, \dots\rangle \\ = (a \cos \theta - b \sin \theta) U^{-1}(R(\theta)) |K, a, b, \dots\rangle$$

$$\text{Let's call } U^{-1}(R(\theta)) |K, a, b, \dots\rangle = |K, a, b, \dots, \theta\rangle$$

$$A |K, a, b, \dots, \theta\rangle = (a \cos \theta - b \sin \theta) |K, a, b, \dots, \theta\rangle$$

$$B |K, a, b, \dots, \theta\rangle = (a \sin \theta + b \cos \theta) |K, a, b, \dots, \theta\rangle$$

There might be infinitely many eigenvalues for different values of θ .

But, as we don't observe such infinitely many eigenvalues from physicist's point of view, we need $a = b = 0$.

For 2 operators which commute, it is necessary that one eigenstate is also the eigenstate of other. But we can always choose. If they are Hermitian, then we can find a complete set of eigenstates. J_1, J_2, J even though they don't commute have same eigenstate $\rightarrow s$ orbital

$$\text{Then, } J_3 |K, \sigma\rangle = \sigma |K, \sigma\rangle$$

$$|p, \sigma\rangle = \sqrt{\frac{K_0}{p_0}} U(L(p)) |K, \sigma\rangle$$

$$W(\alpha, \beta, \theta) = S(\alpha, \beta) R(\theta)$$

$$U(W(\alpha, \beta, \theta)) = e^{i(\alpha A + \beta B)} e^{i\theta J_3}$$

$$U(W(\alpha, \beta, \theta)) |K, \sigma\rangle = e^{i\theta\sigma} |K, \sigma\rangle$$

$$\text{Then, } D_{\sigma\sigma'}(W) = e^{i\theta\sigma} \delta_{\sigma\sigma'}$$

State of the massless particle doesn't mix with any other state

Photon has 2 polarization states, in contrary to 3 spin states is a property of the little group for massless particles.

$\sigma \rightarrow$ Helicity (angular momentum in the direction of momentum)

$$U(\Lambda) |p, \sigma\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} e^{i\sigma \Theta(\Lambda, p)} |\Lambda p, \sigma\rangle$$

Helicity doesn't change under Lorentz transformation.

$$W(\Lambda, p) = L^{-1}(\Lambda p) \Lambda L(p) = S(\alpha(\Lambda, p), \beta(\Lambda, p)) R(\Theta(\Lambda, p))$$

For massive particle, $L(p) = R(\hat{p}) B(|\vec{p}|) R^{-1}(\hat{p})$

Here, we started with $\begin{pmatrix} 0 \\ 0 \\ k \\ k \end{pmatrix}$, so the first rotation is not necessary. We can directly apply the boost, $B\left(\frac{|\vec{p}|}{k}\right)$

$$\begin{pmatrix} 0 \\ 0 \\ k \\ k \end{pmatrix} \xrightarrow{B} \begin{pmatrix} 0 \\ 0 \\ 1 \\ |\vec{p}| \end{pmatrix} \text{ then apply } R(\hat{p}).$$

$$\gamma = \frac{u^2 + 1}{2u}, \quad \beta\gamma = \frac{u^2 - 1}{2u}, \quad \vec{u} = \frac{|\vec{p}|}{k}$$

$$L(p) = R(\hat{p}) B\left(\frac{|\vec{p}|}{k}\right)$$

$$\hat{p} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta) \text{ [chosen]}$$

First rotate about y axis Θ amount and then rotate about z axis (amount $\rightarrow \phi$).

$$R(\hat{p}) = R_3(\phi) R_2(\Theta)$$

$$L(\vec{p}) \vec{k} = \vec{p}, \quad |\vec{p}, \sigma\rangle = \sqrt{\frac{k^0}{p^0}} U(L(\vec{p})) |k, \sigma\rangle.$$

$$U(\Lambda) |\vec{p}, \sigma\rangle = \sqrt{\frac{(\Lambda \vec{p})^0}{p^0}} \sum_{\sigma'} D_{\sigma\sigma'}(W(\Lambda(\vec{p}))) |\Lambda \vec{p}, \sigma'\rangle$$

$$W(\Lambda, \vec{p}) = L^{-1}(\Lambda \vec{p}) \Lambda L(\vec{p})$$

For $M=0$, $K = \begin{pmatrix} 0 \\ 0 \\ 0 \\ M \end{pmatrix}$ $W = SO(3)$.

$$U(\Lambda) |\vec{p}, \sigma\rangle = \sqrt{\frac{(\Lambda \vec{p})^0}{p^0}} \sum_{\sigma'} D_{\sigma\sigma'}^{(j)}(W(\Lambda, \vec{p})) |\Lambda \vec{p}, \sigma'\rangle$$

Start with a state of particular J_z eigenvalue state. Apply a Lorentz transformation and you will get superposition of other J_z eigenvalues. All different σ values mix. All of them has to be $L(\vec{p}) = R(\hat{\vec{p}}) B(|\vec{p}|) R^{-1}(\hat{\vec{p}})$ thought of states of the same particle.

For $M=0$, $K = \begin{pmatrix} 0 \\ 0 \\ K \\ K \end{pmatrix}$ $W = ISO(2)$.

$$U(\Lambda) |\vec{p}, \sigma\rangle = \sqrt{\frac{(\Lambda \vec{p})^0}{p^0}} e^{i\theta} |\Lambda \vec{p}, \sigma\rangle$$

Start with a particular σ value. Apply Lorentz transformation. The momentum will of course change but σ will not. No mixing among σ values.

$$W(\alpha, \beta, \theta) = S(\alpha, \beta) R(\theta)$$

$$L(\vec{p}) = R(\hat{\vec{p}}) B(|\vec{p}|/k)$$

Parity and Time Reversals are Lorentz transformation but not connected to Identity.

Parity $\rightarrow \mathcal{P}^\mu_\nu = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & -1 \end{pmatrix}$ The interval is invariant (same).

Time Reversal $\rightarrow \mathcal{T}^\mu_\nu = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & -1 \end{pmatrix}$

General rule for combining Lorentz-transformations

$$(A, a)(\bar{A}, \bar{a}) = (A\bar{A}, A\bar{a} + a)$$

The states in the Hilbert space would be changed by some operator (as the observer is changed)

$$U(A, a) U(\bar{A}, \bar{a}) = U(A\bar{A}, A\bar{a} + a)$$

(How you get from one observer to another whether in one or several steps do not matter).

$$P = U(\mathcal{P}, 0) \quad T = U(\mathcal{T}, 0)$$

Can we find P and T , such that they obey the above rule? This is not correct for some systems. But we will assume that it is true.

$$\begin{aligned} P U(A, a) P^{-1} &= U(\mathcal{P}, 0) (A, a) (\mathcal{P}, 0)^{-1} \\ &= U(P A P^{-1}, P a) \end{aligned}$$

If you plug in $U(1, a)$ is an infinitesimal transformation, then,

$$P i J^{\mu\nu} P^{-1} = i P^{\mu}_{\rho} P^{\nu}_{\sigma} J^{\rho\sigma}$$

$$P i P^{\mu} P^{-1} = i P^{\mu}_{\rho} P^{\rho}$$

↑ momentum operation
↑ Parity operation

For Time Reversal, $T i J^{\mu\nu} T^{-1} = i J^{\mu}_{\rho} J^{\nu}_{\sigma} J^{\rho\sigma}$

$$T i P^{\mu} T^{-1} = i J^{\mu}_{\rho} P^{\rho}$$

Parity and Time reversal are discrete transformations. Can't be done from identity in continuous steps. So, you don't know if P or T are unitary or anti-unitary.

$$P i H P^{-1} = i H$$

($P^0 = H$ and on R.H.S. only
 $P = 0$ has value as $\mu = 0$ is already chosen $\Rightarrow P^0 = 1$)

If unitary then, i comes out as i only.

$$\Rightarrow P H P^{-1} = H$$

If anti-unitary i comes out as $-i$, then,

$$\Rightarrow P H P^{-1} = -H$$

Similarly for Time reversal,

$$T i H T^{-1} = -i H$$

$$T H T^{-1} = -H$$

$$T H T^{-1} = H$$

Unitary,
Anti- μ ,

$$H|\psi\rangle = E|\psi\rangle.$$

$$\begin{aligned} HP^{-1}|\psi\rangle &= P^{-1}P \cdot HP^{-1}|\psi\rangle. \quad (\text{multiplying } P^{-1}P \text{ in front which is } \mathbb{I}) \\ &= P^{-1}(-H)|\psi\rangle \quad (\text{If anti-unitary}) \\ &= -E P^{-1}|\psi\rangle. \end{aligned}$$

Then corresponding to every eigenstate $|\psi\rangle$ with eigenvalue E , there exists another eigenstate $P^{-1}|\psi\rangle$ with eigenvalue $-E$. So, symmetric spectrum. For you can go as low (to the negative scale) as you want in the energy scale. That is not possible physically.

So, these operators can't be anti-unitary. Parity From same argument T can't be unitary.

Unitary operator $\rightarrow \|U|\psi\rangle\| = \||\psi\rangle\|$.

$$U(a|\psi\rangle + b|\phi\rangle) = aU|\psi\rangle + bU|\phi\rangle$$

Anti-linear Unitary $\rightarrow \|U|\psi\rangle\| = \||\psi\rangle\|$.

$$U(a|\psi\rangle + b|\phi\rangle) = a^*U|\psi\rangle + b^*U|\phi\rangle$$

For linear operators

$(\langle\psi|A)|\phi\rangle = \langle\psi|(A|\phi\rangle)$, these 2 A are not members of same vector space but we have the fact that corresponding to every operator A in a vector space, we have another operator A (labeled the same) in its dual vector space. For our linear cases, they are the same. But not so much for anti-linear cases. So, instead of Dirac's notations, we should use the old notations.

Parity

Now, as the operators should be unitary,

$$P J^{\mu\nu} P^{-1} = J^{\mu\nu} P_e^{\mu\nu} P_e^{\nu\mu} J^{\mu\nu}$$

And, $P J^{\mu\nu} P^{-1} = J^{\mu\nu} P_e^{\mu\nu} P_e^{\nu\mu}$

$$\Rightarrow P \vec{J} P^{-1} = \vec{J} \quad \text{or} \quad T \vec{J} T^{-1} = -\vec{J}$$

$$P \vec{K} P^{-1} = -\vec{K} \quad \text{or} \quad T \vec{K} T^{-1} = \vec{K}$$

$$P \vec{P} P^{-1} = -\vec{P} \quad \text{or} \quad T \vec{P} T^{-1} = -\vec{P}$$

$$P H P^{-1} = H \quad \text{or} \quad T H T^{-1} = H$$

10 generators

9/3/16 $\sqrt{\frac{K^0}{p^0}} U(L(p)) |K, \sigma\rangle = |p, \sigma\rangle$

$$U(\Lambda) |p, \sigma\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'} D_{\sigma'\sigma}(\Lambda(p)) |\Lambda p, \sigma'\rangle$$

For $M > 0$, $(2j+1)$ states. (σ values)

$$K = \begin{pmatrix} 0 \\ 0 \\ 0 \\ M \end{pmatrix} \quad L(p) = R(p) B(|\vec{p}|) R^{-1}(p)$$

$$L(p)^i_j = \delta^i_j + (\gamma - 1) \hat{p}_i \hat{p}_j$$

$$P K |p, \sigma\rangle = P^M |p, \sigma\rangle$$

$$L(p)^0_i = L(p)^i_0 = \sqrt{\gamma^2 - 1} \hat{p}_i$$

$$L(p)^0_0 = \gamma$$

The eigenvalues of $\vec{p} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$H \rightarrow M$$

$$J_3 \rightarrow \sigma$$

$$\vec{P} P |K, \sigma\rangle = -P \vec{P} |K, \sigma\rangle \quad [\text{know } P \vec{P} P^{-1} = -\vec{P}]$$

Parity

	$ k, \sigma\rangle$	$P k, \sigma\rangle$	$T k, \sigma\rangle$
$\vec{P}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
H	M	M	M
J_3	σ	σ	$-\sigma$

Parity does not change σ , Time reversal does σ .

$P|k, \sigma\rangle$ is an eigenvector of $\vec{P}$, H , J_3 with same eigenvalues as of $|k, \sigma\rangle$.

$$P|k, \sigma\rangle = \eta_\sigma |k, \sigma\rangle$$

↑ phase factor depending on σ .

$$T|k, \sigma\rangle = \zeta_\sigma |k, -\sigma\rangle$$

$$(J_1 \pm iJ_2) |k, \sigma\rangle = \sqrt{(j \mp \sigma)(j \pm \sigma + 1)} |k, \sigma \pm 1\rangle$$

(σ is just like m of the ang. mom. algebra)

$$P(J_1 \pm iJ_2) |k, \sigma\rangle = P \sqrt{(j \mp \sigma)(j \pm \sigma + 1)} |k, \sigma \pm 1\rangle$$

$$(J_1 \pm iJ_2) P |k, \sigma\rangle = \sqrt{(j \mp \sigma)(j \pm \sigma + 1)} P |k, \sigma \pm 1\rangle$$

[As $\vec{J}$ and P commutes]

From this algebra, $\eta_\sigma = \eta_{\sigma \pm 1} = \eta$

So, this phase factor cannot be different for different σ .

$$P|k, \sigma\rangle = \eta|k, \sigma\rangle$$

$$P|p, \sigma\rangle = \sqrt{\frac{M}{p^0}} P U(L(p)) |k, \sigma\rangle \quad \text{Here } k^0 = M$$

$$= \sqrt{\frac{M}{p^0}} P U(L(p)) P^{-1} P|k, \sigma\rangle$$

$$= \eta \sqrt{\frac{M}{p^0}} U(P L(p) P^{-1}) |k, \sigma\rangle$$

$$[P L(p) P^{-1}]^i_j = [L(p)]^i_j \quad \left[\begin{array}{l} \text{Just multiplication} \\ \text{of diagonal matrices} \end{array} \right]$$

So, -1 from P and
-1 from P^{-1}

$$[P L(p) P^{-1}]^i_0 = [L(p)]^i_0 = -\sqrt{x^2 - 1} \hat{p}_i$$

$$P L(p) P^{-1} = L(Pp)$$

$$Pp = (-\vec{p}, \sqrt{\vec{p}^2 + M^2})$$

$$\begin{aligned} \text{Then, } P|p, \sigma\rangle &= \eta \sqrt{\frac{M}{p^0}} U(L(Pp)) |k, \sigma\rangle \\ &= \eta |Pp, \sigma\rangle. \end{aligned}$$

So, for massive particle, the parity operation has just changed the momentum to parity reversed momentum. σ is not affected as expected.

Now, for time reversal,

$$T(J_1 \pm iJ_2) |k, \sigma\rangle = T \sqrt{(j \mp \sigma)(j \pm \sigma + 1)} |k, \sigma \pm 1\rangle$$

$$(-J_1 \pm iJ_2) \cdot T |k, \sigma\rangle = \sqrt{(j \mp \sigma)(j \pm \sigma + 1)} T |k, \sigma \pm 1\rangle$$

Here $\vec{J}$ and $\vec{T}$ anticommutes so we get the minus sign in front of $\vec{J}_1$ but not in front of $\vec{J}_2$. As it is antilinear and i becomes $-i$.

$$\text{Then, } (-J_1 \pm iJ_2) \zeta_{\sigma} |K, \sigma\rangle = \sqrt{(j \mp \sigma)(j \pm \sigma + 1)} \zeta_{\sigma \mp 1} |K, -\sigma \mp 1\rangle$$

$$-J_+ \zeta_{\sigma} |K, -\sigma\rangle = \sqrt{(j + \sigma)(j + \sigma + 1)} \zeta_{\sigma+1} |K, -\sigma+1\rangle$$

$$\Rightarrow [\zeta_{\sigma} = -\zeta_{\sigma \pm 1}] = (-1)^{j-\sigma} \zeta$$

$$(*) \quad |K, \sigma^*\rangle = \zeta^{1/2} |K, \sigma\rangle$$

$$\begin{aligned} T |K, \sigma^*\rangle &= \zeta^{*1/2} T |K, \sigma\rangle \\ &= \zeta^{*1/2} (-1)^{j-\sigma} \zeta |K, -\sigma\rangle \\ &= (-1)^{j-\sigma} \zeta^{1/2} |K, -\sigma\rangle \\ &= (-1)^{j-\sigma} |K, -\sigma\rangle' \end{aligned}$$

For anti-linear operation, you can get rid of ζ by redefining the states.

$$T |K, \sigma\rangle = (-1)^{j-\sigma} \zeta |K, -\sigma\rangle$$

$$T |p, \sigma\rangle = (-1)^{j-\sigma} \zeta |p, -\sigma\rangle \leftarrow (\text{guess, not right})$$

$$T |p, \sigma\rangle = \sqrt{\frac{M}{p_0}} T U(L(p)) |K, \sigma\rangle$$

$$= \sqrt{\frac{M}{p_0}} U(\gamma L(p) \gamma^{-1}) T |K, \sigma\rangle$$

$$= (-1)^{j-\sigma} \sum \sqrt{\frac{M}{p_0}} U(L(p)) |K, -\sigma\rangle$$

$$= (-1)^{j-\sigma} \sum |Pp, -\sigma\rangle$$

10/3/16 for $M=0$

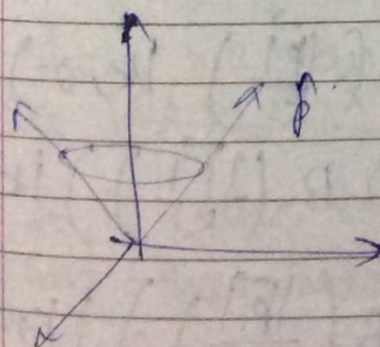
$$K = \begin{pmatrix} 0 \\ 0 \\ K \\ K \end{pmatrix} \quad L(p) = R(p) B \left(\frac{|P^2|}{K} \right)$$

$$|p, \sigma\rangle = \sqrt{\frac{K}{p_0}} U(L(p)) |K, \sigma\rangle$$

$$R(p) = e^{i\theta J_3} e^{i\phi J_2} \quad \text{where } \hat{p} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$$

$$U(\Lambda) |p, \sigma\rangle = \sqrt{\frac{(\Lambda p)_0}{p_0}} e^{i\sigma\theta(\Lambda p)} |\Lambda p, \sigma\rangle$$

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{u^2+1}{2u} & \frac{u^2-1}{2u} \\ 0 & 0 & \frac{u^2-1}{2u} & \frac{u^2+1}{2u} \end{pmatrix}$$



	$ K, \sigma\rangle$	$P K, \sigma\rangle$	$T K, \sigma\rangle$
$\vec{p}$	$\begin{pmatrix} 0 \\ 0 \\ K \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ -K \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ -K \end{pmatrix}$
$\hat{H}$	K	K	K

$U(R_2) = e^{-i\pi J_2}$ J_3 σ σ $-\sigma$
To change from K to $-K$, a rotation of " π " around x or y axis

T changes the sign of t and leaves the spatial coordinates unchanged. It changes the sign of velocity through $(\frac{dy}{dt})$, so the momentum.

$$PT |k, \sigma\rangle = -TP |k, \sigma\rangle = -\begin{pmatrix} 0 & 0 \\ 0 & +k \end{pmatrix} T |k, \sigma\rangle$$

	$U(R_2^{-1}) P k, \sigma\rangle$	$U_2(R_2^{-1}) T k, \sigma\rangle$
$\vec{p}$	$\begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}$
H	k	k
J_3	$-\sigma$	σ

J_3 is the 3rd component, under R_2 notation, it will change sign.

$$U(R_2^{-1}) P |k, \sigma\rangle = \eta_\sigma |k, -\sigma\rangle$$

$$U(R_2^{-1}) T |k, \sigma\rangle = \xi_\sigma |k, \sigma\rangle$$

We know that,

$$|p, \sigma\rangle = \sqrt{\frac{k}{p_0}} U(R(\hat{p})) B\left(\frac{|\vec{p}|}{k}\right) |k, \sigma\rangle$$

$$P |p, \sigma\rangle = \sqrt{\frac{k}{p_0}} U(P R(\hat{p})) B\left(\frac{|\vec{p}|}{k}\right) |k, \sigma\rangle$$

$$= \sqrt{\frac{k}{p_0}} \left\{ U(R(\hat{p})) P B\left(\frac{|\vec{p}|}{k}\right) \right\} |k, \sigma\rangle$$

(Rotation and parity commutator)

Rotation is a 3×3 matrix, Parity $\rightarrow 4 \times 4$
But its main action is restricted to 3×3 part
But boost is a 4×4 matrix. We can't move P across it so easily.

Boost can be written as a 2×2 matrix
 $\rightarrow 2 \oplus 2$ block matrix decomposition.

Now parity can be written as a $3 \oplus 1$ blocks. But if you do R_2^{-1} , then the z component and x component becomes $+$.

$$\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \rightarrow \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$$

If we do $R_2^{-1} P$, then its 2 by 2 blocks will commute with the 2×2 blocks of the boost.

$$\begin{aligned} P(p, \sigma) &= \sqrt{\frac{K}{p_0}} U(R(\hat{p}) R_2 R_2^{-1} P B\left(\frac{|\vec{p}|}{K}\right)) |k, \sigma\rangle \\ &= \eta_\sigma \sqrt{\frac{K}{p_0}} U(R(\hat{p}) R_2 B\left(\frac{|\vec{p}|}{K}\right)) |k, -\sigma\rangle \end{aligned}$$

If $R(\hat{p}) R_2$ can be replaced by $R(-\hat{p})$
Then, we can write $U(R(-\hat{p}) B\left(\frac{|\vec{p}|}{K}\right))$
as $U(P L(\hat{p})) U(L(p \hat{p}))$.

It is same as the spatial parts rotated and then acted upon by Parity.

If $\hat{p} = (\theta, \phi)$, $-\hat{p} = (\pi - \theta, \phi \pm \pi)$

$$0 \leq \theta \leq \pi$$

$$0 \leq \phi \leq 2\pi$$

$$\phi' = \begin{cases} \phi + \pi & \text{if } 0 \leq \phi < \pi \\ \phi - \pi & \text{if } \pi \leq \phi < 2\pi \end{cases}$$

$$\eta_\sigma \sqrt{\frac{k}{p_0}} U(R(-\hat{p}) B(\frac{|\vec{p}|}{k})) |k, -\sigma\rangle$$

$$= \eta_\sigma \sqrt{\frac{k}{p_0}} U(L(\hat{p})) |k, -\sigma\rangle = \eta_\sigma |p, -\sigma\rangle$$

Let's check what is the difference between, $R(-\hat{p})$ and $R_2 R(\hat{p})$.

$$U(R(\hat{p})) = e^{i\phi J_3} e^{i\theta J_2}$$

$$U^{-1}(R(-\hat{p})) U(R(\hat{p}) R_2) \quad (\text{If these 2 are same, then we should get } \mathbb{I})$$

$$= [e^{i(\phi \pm \pi) J_3} e^{i(\pi - \theta) J_2}]^{-1} e^{i\phi J_3} e^{i\theta J_2} e^{-i\pi J_2}$$

$$= e^{-i(\theta - \pi) J_2} e^{-i(\phi \pm \pi) J_3} e^{i\phi J_3} e^{i(\theta - \pi) J_2}$$

$$= e^{i(\theta - \pi) J_2} e^{-i(\theta - \pi) J_2} e^{-i J_3 \pi}$$

(J_2 and J_3 don't commute, but there is a " π " rotation around J_3 axis, it just changes the sign of J_2)

$$= e^{-i J_3 \pi}$$

$$\text{Then, } |p, \sigma\rangle = \eta_\sigma U(R(-\hat{p})) e^{-i\pi J_3} U(B(\frac{|\vec{p}|}{k})) |k, \sigma\rangle$$

(The rotation about J_3 can easily jump across boost.

as the non-trivial action is along 2 and 3

$$= \eta_\sigma e^{\pm i\pi\sigma} \sqrt{\frac{k}{p_0}} U(L(\hat{p})) |k, -\sigma\rangle = \eta_{\sigma\mp} |p, -\sigma\rangle$$

For $M=0$,

$$\begin{array}{ccc}
 |k, \sigma\rangle & T|k, \sigma\rangle & U(R_2^{-1})T|k, \sigma\rangle \\
 p^M \begin{pmatrix} 0 \\ 0 \\ k \\ k \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ -k \\ k \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ k \\ k \end{pmatrix} \\
 J_3 & \sigma & -\sigma & \sigma
 \end{array}$$

$$U_2(R_2^{-1})T|k, \sigma\rangle = \mathbb{J}_\sigma |k, \sigma\rangle$$

$$|p, \sigma\rangle = \sqrt{\frac{k}{p^0}} U(L(p)) |k, \sigma\rangle$$

$$L(p) = R(\hat{p}) B\left(\frac{|\vec{p}|}{k}\right)$$

Now, the question is what T does when acting on $|p, \sigma\rangle$.

$$T|p, \sigma\rangle = \sqrt{\frac{k}{p^0}} U(R(\hat{p})) \mathbb{J} B\left(\frac{|\vec{p}|}{k}\right) |k, \sigma\rangle$$

$\mathbb{J}$ and $R(\hat{p})$ can commute as,

$$R(\hat{p}) = \left(\begin{array}{c|c} \times & 0 \\ \hline 0 & 1 \end{array} \right) \text{ and } \mathbb{J} = \left(\begin{array}{c|c} \mathbb{I} & 0 \\ \hline 0 & -1 \end{array} \right)$$

Not true for $\mathbb{J}$ and $B\left(\frac{|\vec{p}|}{k}\right)$.

$$\left(\begin{array}{c|cc} \mathbb{I} & 0 & \\ \hline 0 & \frac{u^2+1}{2u} & \frac{u^2-1}{2u} \\ & \frac{u^2-1}{2u} & \frac{u^2+1}{2u} \end{array} \right) \left(\begin{array}{c|cc} 1 & 0 & \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ & 0 & 0 & -1 \end{array} \right) \text{ don't commute}$$

Apply R_2^{-1} on $\mathbb{J}$, then, x and z components become -1 , $\mathbb{J}$ of boost commutes with first block of $R_2^{-1}\mathbb{J}$.

And negative identity can commute with the second block of boost. $\rightarrow \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

Then,

$$\begin{aligned} T|p, \sigma\rangle &= \sqrt{\frac{k}{p_0}} U(R(\hat{p}) R_2 R_2^{-1} J B\left(\frac{|\vec{p}|}{k}\right)) |k, 0\rangle \\ &= \sqrt{\frac{k}{p_0}} U(R(\hat{p}) R_2 B\left(\frac{|\vec{p}|}{k}\right)) U(R_2^{-1}) T|k, 0\rangle \end{aligned}$$

Let's check if we can prove that $U(R(\hat{p}) R_2)$ can be replaced by $U(R(-\hat{p}))$.

$$\begin{aligned} &U^{-1}(R(-\hat{p})) U(R(\hat{p}) R_2) \quad [R_3 J_2 R_3^{-1} = -J_2] \\ &= e^{-i(\pi-\theta)J_2} e^{\mp i\pi J_3} e^{-i(\pi-\theta)J_2} \\ &= e^{\mp i\pi J_3} \quad (\text{In Weinberg, it is given, } e^{\pm i\pi J_3}, \text{ check}) \end{aligned}$$

Then, $U(R(\hat{p}) R_2) = U(R(-\hat{p})) e^{\mp i\pi J_3}$.

In case of parity, $P|p, \sigma\rangle = \eta_\sigma e^{\pm i\pi\sigma} |\mathcal{P}p, \sigma\rangle$

In this case,

$$\begin{aligned} T|p, \sigma\rangle &= \sqrt{\frac{k}{p_0}} U(R(-\hat{p}) B\left(\frac{|\vec{p}|}{k}\right)) e^{\mp i\pi J_3} \sum_{\sigma} |k, \sigma\rangle \\ &= \sum_{\sigma} e^{\mp i\pi\sigma} U(L(\hat{p})) |k, \sigma\rangle \\ &= \sum_{\sigma} e^{\mp i\pi\sigma} |\mathcal{P}p, \sigma\rangle \end{aligned}$$

The helicity does not change.

$$0 \leq \phi < 2\pi$$

$0 \leq \phi < \pi \rightarrow$ upper sign, $\pi \leq \phi < 2\pi \rightarrow$ lower sign

For $M=0$,

$$T|p, \sigma\rangle = (-1)^{j-\sigma} |p, -\sigma\rangle$$

$$\begin{aligned} T^2|p, \sigma\rangle &= (-1)^{j-\sigma} (-1)^{j+\sigma} |p, \sigma\rangle \\ &= (-1)^{2j} |p, \sigma\rangle \end{aligned}$$

For $M=0$, $T^2|p, \sigma\rangle = (-1)^{2|0|} |p, \sigma\rangle$

$T^2|\psi\rangle = -|\psi\rangle$ for odd no. of particles.

T, H commutes, if T and H commute then $T|\psi\rangle$ is an eigenstate of H as $|\psi\rangle$ is.

If $T|\psi\rangle$ is a multiple of $|\psi\rangle$, then,

$$T^2|\psi\rangle \neq -|\psi\rangle$$

Then, $T|\psi\rangle \neq \pm |\psi\rangle$

Time reversal symmetry in odd no. of fermion system $\rightarrow$ every state is doubly degenerate.

$$\Rightarrow |\psi\rangle \text{ and } T|\psi\rangle$$

16/3/16 Gauge Symmetry :

Real Klein-Gordon field $\rightarrow$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \quad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\Rightarrow \boxed{\partial_\mu \partial^\mu \phi + m^2 \phi = 0}$$

Two real Klein-Gordon fields with equal mass
 $\mathcal{L} = \frac{1}{2} [\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2] - \frac{m^2}{2} (\phi_1^2 + \phi_2^2)$

Let's take 2 complex fields,

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \quad \phi^* = \frac{\phi_1 - i\phi_2}{\sqrt{2}}$$

$$\text{Then, } \mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi^* \phi$$

We can start with either ϕ_1, ϕ_2 or ϕ, ϕ^* .

$$\delta \mathcal{L} = A \delta \phi_1 + B \delta \phi_2 \quad A = \frac{\delta \mathcal{L}}{\delta \phi_1} \rightarrow \text{functional derivative}$$

$$= A \cdot \left(\frac{\delta \phi + \delta \phi^*}{\sqrt{2}} \right) + B \left(\frac{\delta \phi - \delta \phi^*}{\sqrt{2}i} \right)$$

$$= \left(\frac{A + iB}{\sqrt{2}} \right) \delta \phi^* + \left(\frac{A - iB}{\sqrt{2}} \right) \delta \phi$$

$\Rightarrow \phi$ and ϕ^* are also be treated as independent variables.

$$A = B = 0 \Rightarrow A + iB = A - iB = 0$$

(Linear independence)

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi^* \phi$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = \partial^\mu \phi^*$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^*$$

$$\Rightarrow \partial_\mu \partial^\mu \phi^* + m^2 \phi^* = 0$$

$$\text{And, } \partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

$$\begin{cases} (\square + m^2) \phi_1 = 0 \\ (\square + m^2) \phi_2 = 0 \end{cases}$$

Suppose, you change the phase of ϕ ,

$$\phi \rightarrow e^{-i\Lambda} \phi, \quad \phi^* \rightarrow e^{i\Lambda} \phi^*$$

$$\text{Then, } \phi_1 \rightarrow \phi_1 \cos \Lambda + \phi_2 \sin \Lambda$$

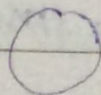
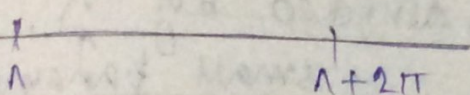
$$\phi_2 \rightarrow -\phi_1 \sin \Lambda + \phi_2 \cos \Lambda$$

In complex plane, we just multiply by a phase factor. $\Rightarrow U(1) \rightarrow$ group structure

In real plane, it seems like that ϕ_1, ϕ_2 get mixed up. $\rightarrow$ like rotation in real plane.

$\Rightarrow SO(2) \rightarrow$ group structure.

These 2 are isomorphic.



1 and $1+2\pi$ have similar effects.

$\Rightarrow$ circle.

Global gauge symmetry

$\Rightarrow$ continuous symmetry $\rightarrow \lambda$ takes values continuously. $\rightarrow$ continuous symmetry has an associated conserved quantity.

Continuous symmetry parametrized by, $w^1, w^2, \dots, w^r$.

$$\partial_\mu J^\mu_a = 0 \Rightarrow \int J^\mu_a d^3x = Q_a \quad \text{is locally conserved}$$

For infinitesimal transformation,

$$\Delta x^\nu = x^\nu_a \delta w^a$$

$$\Delta \phi = \phi_a \delta w^a = \phi'(x) - \phi(x)$$

$$\delta \phi = \phi'(x) - \phi(x)$$

$$\text{Then, } J^\mu_a = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Phi_a - \Theta^\mu_\nu x^\nu_a$$

$$\Rightarrow \Theta^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}$$

What happens if we have more than one field?
Then it should get summed up over index i .

$$J^\mu_a = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \Phi_{ia} - \Theta^\mu_\nu x^\nu_a$$

$$\Theta^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i - \delta^\mu_\nu \mathcal{L}$$

Here, $\chi^V = 0$. (only phase factor changes, not the coordinates)

$$\phi \rightarrow e^{-i\Lambda} \phi \simeq \phi - i\Lambda \phi$$

$$\phi^* \rightarrow e^{i\Lambda} \phi^* \simeq \phi^* + i\Lambda \phi^*$$

$\Phi \rightarrow$ change in ϕ divided by Λ .
small parameter.

$$\underline{\Phi} = -i\phi$$

$$\underline{\Phi}^* = +i\phi^*$$

$$\begin{aligned} \text{Then, } \mathcal{J}^\mu &= (\partial_\mu \phi^*)(-i\phi) + (\partial_\mu \phi)(i\phi^*) \\ &= i(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*) \end{aligned}$$

$$\partial_\mu \mathcal{J}^\mu = i\phi^* \square \phi - i\phi \square \phi^* = 0.$$

$\rightarrow$ Equation of Motion

$$\left. \begin{aligned} \phi &\rightarrow e^{-i\Lambda(x)} \phi \\ \phi^* &\rightarrow e^{i\Lambda(x)} \phi^* \end{aligned} \right\} \text{Not a symmetry of } \mathcal{L}$$

For infinitesimal transformation,

$$\delta\phi = -i\Lambda\phi, \quad \delta\phi^* = i\Lambda\phi^*$$

$$(\partial_\mu \phi) \rightarrow \partial_\mu(\phi - i\Lambda\phi) = \partial_\mu \phi - i\Lambda \partial_\mu \phi - i(\partial_\mu \Lambda)\phi$$

$$\delta(\partial_\mu \phi) = -i\Lambda(\partial_\mu \phi) - i(\partial_\mu \Lambda)\phi$$

$$\delta(\partial_\mu \phi^*) = i\Lambda(\partial_\mu \phi^*) + i(\partial_\mu \Lambda)\phi^*$$

How does the Lagrangian change under this transformation?

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi^* \phi$$

$$\delta \mathcal{L} = [\delta(\partial_\mu \phi)] \partial^\mu \phi^* + \partial_\mu \phi [\delta(\partial^\mu \phi^*)] - m^2(\delta \phi^* \phi + \phi^* \delta \phi)$$

$$\neq [-i\cancel{\Lambda} \partial_\mu \phi - i(\partial_\mu \Lambda) \phi] \partial^\mu \phi^* + \partial_\mu \phi [i\cancel{\Lambda} \partial^\mu \phi^* + i(\partial^\mu \Lambda) \phi^*] - m^2(i\cancel{\Lambda} \phi^* \phi + \phi^* (-i\cancel{\Lambda} \phi))$$

$$= (\partial_\mu \Lambda) J^\mu$$

The variation in Lagrangian is not zero anymore. So, this is not a symmetry of the Lagrangian.

To make this $\mathcal{L}$ symmetric we can add subtract $\partial_\mu \phi^* \partial^\mu \phi$, so that the troublesome piece goes away.

⊗ We can add some scalar quantities to the Lagrangian, to make it locally gauge symmetric.
 ϕ, ϕ^*, A_μ $\mathcal{L} + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \dots$
 $A_\mu \rightarrow A_\mu + \delta A_\mu$ $\mathcal{L}_1 = -e A_\mu J^\mu$
 $\mathcal{L}_2 = \frac{1}{2} A_\mu A^\mu$

$\mathcal{L}_3$ should be a derivative term of A_μ , otherwise it will not be a differential eq.

17/3/16 $\mathcal{L}_0 = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^*$

Under the transformation, $\phi \rightarrow e^{-i\Lambda} \phi$
 $\phi^* \rightarrow e^{i\Lambda} \phi^*$

$\delta \mathcal{L}_0 = 0$ if Λ is a constant.

The infinitesimal version,

$$\delta \phi = -i\Lambda \phi \quad \text{and} \quad \delta \phi^* = +i\Lambda \phi^*$$

If Λ is a function of x , then,
we obtained, $\delta \mathcal{L}_0 = (\partial_\mu \Lambda) J^\mu$.

We would like to find out a new Lagrangian
which is gauge invariant.

$A_\mu \rightarrow$ a vector field. $A_\mu \rightarrow A'_\mu = A_\mu + \delta A_\mu$

The Lagrangian $\rightarrow$ Lorentz scalar

Terms to be added,

$$L_1 = -e A_\mu J^\mu \quad L_2 = \frac{1}{2} A_\mu A^\mu$$

$$\delta J^\mu = \delta (i \phi^* \partial_\mu \phi - i \phi \partial_\mu \phi^*) = 2 \delta [\text{Im}(\phi \partial_\mu \phi^*)]$$

$$= 2 \text{Im} [\delta \phi \partial_\mu \phi^* + \phi \delta (\partial_\mu \phi^*)]$$

$$= 2 \text{Im} [-i \Lambda \phi \partial_\mu \phi^* + \phi (i \Lambda \partial_\mu \phi^* + i (\partial_\mu \Lambda) \phi^*)]$$

$$= 2 |\phi|^2 (\partial_\mu \Lambda)$$

The other terms should get cancelled, as
for $\Lambda = \text{constant}$ $\partial_\mu \Lambda = 0$ and $\delta \mathcal{L} = 0$.

$$\text{Then, } \delta \mathcal{L}_1 = -e (\delta A_\mu) J^\mu - 2e A_\mu |\phi|^2 (\partial^\mu \Lambda)$$

$$\delta \mathcal{L}_2 = -2 \int A_\mu \delta A^\mu$$

Total variation, $\delta \mathcal{L}_0 + \delta \mathcal{L}_1 + \delta \mathcal{L}_2$

$$= (\partial_\mu \Lambda) J^\mu - e (\delta A_\mu) J^\mu - 2e A_\mu |\phi|^2 (\partial^\mu \Lambda) + 2 \int A_\mu \delta A^\mu$$

For the first 2 terms to get cancelled, we
can write, $\delta A_\mu = \frac{1}{e} (\partial_\mu \Lambda)$.

Then, for the last 2 pieces, $\frac{1}{2} = e|\phi|^2$

$$\Rightarrow \mathcal{L}_1 = -e A_\mu J^\mu \quad \text{and} \quad \mathcal{L}_2 = e^2 |\phi|^2 A_\mu A^\mu.$$

$$\Rightarrow A_\mu \rightarrow A'_\mu = A_\mu + \frac{1}{e} (\partial_\mu \Lambda).$$

$\Rightarrow$ You get gauge invariance, in this case.

We want to construct something which is gauge invariant and derivative of A_μ .
(To know, how A_μ will evolve in time)

$$\begin{aligned} \Rightarrow \partial_\mu A_\nu &\rightarrow \partial_\mu A'_\nu = \partial_\mu \left(A_\nu + \frac{1}{e} \partial_\nu \Lambda \right) \\ &= \partial_\mu A_\nu + \frac{1}{e} \partial_\mu \partial_\nu \Lambda. \end{aligned}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \rightarrow \text{gauge invariant.} \quad \boxed{F'_{\mu\nu} = F_{\mu\nu}}.$$

The third piece of Lagrangian, should be quadratic in derivatives (scalar) and gauge invariant.

$$\Rightarrow \mathcal{L}_3 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

(This is the same Lagrangian of that of a charged particle in electromagnetic field).

The full Lagrangian,

$$\begin{aligned} \mathcal{L} = & \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi^* \phi - ie A_\mu (\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) \\ & + e^2 A_\mu A^\mu \phi^* \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \end{aligned}$$

$$= (\partial_\mu \phi + ie A_\mu \phi) (\partial^\mu \phi^* - ie A^\mu \phi^*) - m^2 \phi^* \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

$$= (D_\mu \phi) (D^\mu \phi)^* - m^2 \phi^* \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

$$\text{Here, } D_\mu \phi \equiv \partial_\mu \phi + ie A_\mu \phi$$

$\uparrow$
covariant derivative.

$$\begin{aligned}
 \delta(D_\mu \phi) &= \delta [\partial_\mu \phi + ie\phi A_\mu] \\
 &= -i\Lambda \partial_\mu \phi - i(\partial_\mu \Lambda)\phi + ie(-i\Lambda\phi) A_\mu \\
 &\quad + ie\phi \cdot \frac{1}{e} \partial_\mu \Lambda \\
 &= -i\Lambda (D_\mu \phi)
 \end{aligned}$$

$D_\mu \phi$ variation doesn't have any extra piece like $\partial_\mu \phi$, it changes similarly as ϕ does.

Similarly, $\delta(D^\mu \phi)^* = +i\Lambda (D^\mu \phi)^*$
 $\Rightarrow$ Invariant Lagrangian.

$D_\mu \phi \equiv \partial_\mu \phi + ie\phi A_\mu$ (Let's consider that we don't know how A_μ changes)

$$\phi \rightarrow e^{-i\Lambda} \phi$$

$$(D_\mu \phi) \rightarrow (D_\mu \phi)' = [\partial_\mu \phi' + ie\phi' A_\mu']$$

$$\begin{aligned}
 (D_\mu \phi)' &= \partial_\mu (e^{-i\Lambda} \phi) + ie e^{-i\Lambda} \phi A_\mu' \\
 &= e^{-i\Lambda} (\partial_\mu \phi + ie\phi A_\mu) \leftarrow \text{we want it to be this} \\
 &\Rightarrow e^{-i\Lambda} (\partial_\mu \phi - i(\partial_\mu \Lambda)\phi + ie\phi A_\mu')
 \end{aligned}$$

If we want $D_\mu \phi$ to be gauge covariant,

$$\Rightarrow e A_\mu' - \partial_\mu \Lambda = e A_\mu$$

$$(D_\mu \phi)' = e^{-i\Lambda} (D_\mu \phi)$$

$$\Rightarrow \boxed{A_\mu' = A_\mu + \frac{1}{e} \partial_\mu \Lambda}$$

$$A^\mu = (A^0, \vec{A})$$

$$A_\mu = (A^0, -\vec{A})$$

$$(\partial_\mu + ieA_\mu) = D_\mu$$

$$\vec{\nabla} \rightarrow \vec{\nabla} - ie\vec{A} \quad (\text{classical})$$

$$\vec{p} \rightarrow \vec{p} - e\vec{A} \quad (\text{quantum})$$



How a charged Klein-Gordon particle behaves in an electromagnetic field. (Non-Relativistic limit).

$$\begin{aligned}\phi_1' &= \phi_1 \cos \Lambda + \phi_2 \sin \Lambda \\ \phi_2' &= -\phi_1 \sin \Lambda + \phi_2 \cos \Lambda\end{aligned}$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}' = \begin{pmatrix} R_{3 \times 3} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \quad \begin{aligned} \phi &\rightarrow R\phi \\ \partial_\mu \phi &\rightarrow R(\partial_\mu \phi) \\ D_\mu \phi &\rightarrow R(D_\mu \phi) \end{aligned}$$

Build up a Lagrangian depending on the invariants under this transformation.

(*) Work out the equation of motion for this Lagrangian.

28/3/16 Gauge theories: Non-Abelian case.

$$\mathcal{L}_{\text{free}} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^*.$$

$$\phi \rightarrow e^{-i\theta} \phi \quad \text{constant} \quad \Rightarrow \quad \phi \rightarrow e^{-i\theta(x)} \phi$$

$$\partial_\mu \phi \rightarrow e^{-i\theta} \partial_\mu \phi \quad \text{Here, } \partial_\mu \phi \not\rightarrow e^{-i\theta} \partial_\mu \phi$$

$$D_\mu \equiv \partial_\mu - ieA_\mu \quad D_\mu \phi \rightarrow e^{-i\theta} D_\mu \phi$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$\phi \equiv \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix} \quad U \equiv N \times N \text{ matrix.}$$

Global Symmetry: $\phi \rightarrow U\phi$
 $\partial_\mu \phi \rightarrow U \partial_\mu \phi$
 $(\partial_\mu \phi)^\dagger (\partial^\mu \phi) \rightarrow \text{invariant}$
 $\phi^\dagger \phi$

"Gauge" this symmetry.

$$\phi \rightarrow U\phi$$

$$\partial_\mu \phi \rightarrow U \partial_\mu \phi + (\partial_\mu U) \phi$$

$$D_\mu = \partial_\mu - ig A_\mu$$

$$D_\mu \phi \rightarrow U (D_\mu \phi)$$

(A_μ are fields
fnc. of x, t . But they
are matrices).

Here, D_μ should be a 4-vector
 ∂_μ here is ∇ multiplied with ∂_μ .
For a Lorentz transformation, the components
of A^μ ($-A_0, A_1, A_2, A_3$ get mixed up) (?).

$$D_\mu \rightarrow U D_\mu U^{-1} \quad \text{Then, } D_\mu \phi \rightarrow (U D_\mu U^{-1})(U\phi)$$

$$\Rightarrow U(D_\mu \phi)$$

$$\partial_\mu - ig A_\mu \rightarrow \partial_\mu - ig A'_\mu = U(\partial_\mu - ig A_\mu)U^{-1}$$

$$\begin{aligned} U \partial_\mu (U^{-1} \phi) &\rightarrow = U U^{-1} \partial_\mu \phi + U(\partial_\mu U^{-1}) \phi - ig U A_\mu U^{-1} \phi \\ &= U \partial_\mu \phi + U(\partial_\mu U^{-1}) \phi - ig U A_\mu U^{-1} \phi \\ &= \partial_\mu \phi + U(\partial_\mu U^{-1}) \phi - ig U A_\mu U^{-1} \phi \end{aligned}$$

$$\text{Then, } A_\mu \rightarrow A'_\mu = U A_\mu U^{-1} + \frac{i}{g} U(\partial_\mu U^{-1})$$

$$= U A_\mu U^{-1} - \frac{i}{g} (\partial_\mu U) U^{-1}$$

$$\text{If } U = e^{-ig\theta}$$

$$A'_\mu = A_\mu - \frac{i}{g} (ig) e^{-ig\theta} \partial_\mu \theta e^{ig\theta}$$

Gauge Invariant theory $\rightarrow$

$$\phi \rightarrow U\phi$$

$$\partial_\mu \phi \rightarrow U \partial_\mu \phi$$

$$D_\mu \phi \rightarrow U(D_\mu \phi)$$

$$D_\mu = \partial_\mu - i g A_\mu$$

$$A_\mu \rightarrow U A_\mu U^{-1} - \frac{1}{g} (\partial_\mu U) U^{-1}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \leftarrow \text{These are derivatives}$$

$$\text{Quadratic} \rightarrow -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \leftarrow \text{Lorentz invariant}$$

$$\text{As } D_\mu \rightarrow U D_\mu U^{-1}$$

$$D_\mu D_\nu \rightarrow U D_\mu D_\nu U^{-1}$$

$$\text{Then, } [D_\mu, D_\nu] = U [D_\mu, D_\nu] U^{-1}$$

$$[\partial_\mu - i g A_\mu, \partial_\nu - i g A_\nu]$$

$$= -i g [\partial_\mu A_\nu - \partial_\nu A_\mu]$$

$$\left[\frac{\partial}{\partial x}, f(x) \right] g(x)$$

$$= \frac{\partial}{\partial x} (f g) - \frac{\partial g}{\partial x} f$$

$$= \frac{\partial f}{\partial x} g$$

By analogy,

$$-i g F_{\mu\nu} = [D_\mu, D_\nu]$$

$$F_{\mu\nu} \rightarrow U F_{\mu\nu} U^{-1}$$

$$F_{\mu\nu} F^{\mu\nu} \rightarrow U F_{\mu\nu} F^{\mu\nu} U^{-1}$$

$F_{\mu\nu} F^{\mu\nu}$ cannot go into the Lagrangian because it is a matrix. $\mathcal{L}$ has only numbers.

Not Gauge invariant

So, we need to consider trace of these matrices.

For dynamics of A_μ , we need $F_{\mu\nu}$.

Bringing in Lie algebra into the picture,

$$U = e^{-i g \theta^a T_a}$$

$$\simeq \mathbb{I} - i g \theta^a T_a$$

$$[T_a, T_b] = i f_{abc} T_c$$

$$= i f_{abc} T_c$$

[for completely antisymmetric cases]

$$\begin{aligned}
 A_\mu' &= U A_\mu U^{-1} - \frac{i}{g} (\partial_\mu U) U^{-1} \\
 &= A_\mu - ig \Theta^a [T_a, A_\mu] - \frac{i}{g} (-ig \partial_\mu \Theta^a) T_a \\
 &\quad \text{(upto 1st order of } \Theta^a) \\
 &= A_\mu - ig \Theta^b [T_b, A_\mu] - (\partial_\mu \Theta^a) T_a
 \end{aligned}$$

Let's say, $A_\mu = A_\mu^a T_a$ (Linear combination of T_a 's)

$$\begin{aligned}
 A_\mu'^a &= A_\mu^a - (\partial_\mu \Theta^a) - ig f_{bc}^a \Theta^b A_\mu^c \\
 &= A_\mu^a - (\partial_\mu \Theta^a) + g f_{bc}^a \Theta^b A_\mu^c
 \end{aligned}$$

$$\begin{aligned}
 -ig F_{\mu\nu} &= [\partial_\mu - ig A_\mu^b T_b, \partial_\nu - ig A_\nu^c T_c] \\
 &= -ig (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - ig f_{bc}^a A_\mu^b A_\nu^c) T_a
 \end{aligned}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{bc}^a A_\mu^b A_\nu^c$$

For gauging isospins, it is just cross-product

$$\text{Tr}(F_{\mu\nu} F^{\mu\nu}) = F_{\mu\nu}^a F^{\mu\nu b} \text{Tr}(T_a T_b) = K F_{\mu\nu}^a F^{\mu\nu a}$$

This goes into Lagrangian. In $U(1)$ theory, like in electromagnetism, the $F_{\mu\nu}$'s are just numbers and gauge-invariant. But here, they are gauge covariant. But they change in a very simple manner.

30/3/16 $\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 |\phi|^2$ Global gauge sym.

$$\phi \rightarrow e^{-i\theta} \phi \quad A_\mu \rightarrow A_\mu + \frac{1}{e} \partial_\mu \theta$$

$$\mathcal{L} = (\partial_\mu + ie A_\mu) \phi \cdot (\partial^\mu - ie A^\mu) \phi^* - m^2 |\phi|^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Local Gauge sym

Spontaneous Symmetry Breaking

$$f(x) \equiv x^2 - 4 = 0$$

$$\Rightarrow x \rightarrow -x \quad \{2, -2\}$$

Symmetry of eq.

Individual solutions here don't have the symmetry of the equation

In Ising model, $m \rightarrow -m$

$$H = - \sum_{\langle ij \rangle} J s_i s_j - h \sum s_i$$

$\langle ij \rangle$

(symmetric under $s_i \rightarrow -s_i$)

Explicit symmetry breaking by introducing a mag. field

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - V(\phi)$$

Let's define, $V(\phi) = m^2 |\phi|^2 + \lambda |\phi|^4$

This is also invariant under global gauge sym

For a stable ground state, $\lambda > 0$.

$m^2 > 0$ is a possibility. A more interesting possibility is $m^2 < 0$.

Hamiltonian, $H = \pi \dot{\phi} + \pi^* \dot{\phi}^* - \mathcal{L}$

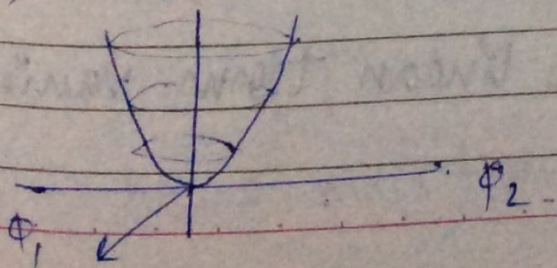
Minimum energy $\phi = \text{constant}$

For $m^2 > 0$, $\boxed{\phi = 0}$

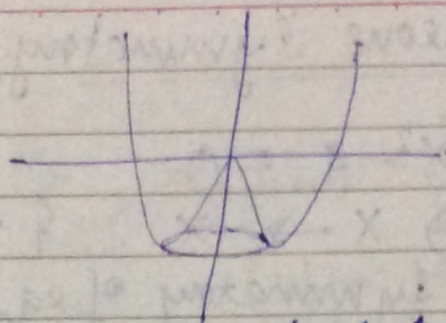
$$\boxed{\langle 0 | \phi | 0 \rangle = 0}$$

Vacuum expectation value

The vacuum state has the symmetry for, $m^2 > 0$.



For $m^2 < 0$,
(Mexican hat)



The minimum energy $\rightarrow$ infinitely possible values $\rightarrow$ infinite vacuum states.

$$V(\phi) = m^2 |\phi|^2 + \lambda |\phi|^4$$

$$\frac{dV}{d\phi} = 2m^2 |\phi| + 4\lambda |\phi|^3$$

Then, $|\phi|_0 = 0$ or $|\phi|^2 = \frac{-m^2}{2\lambda}$ for $m^2 < 0$.

$$\Rightarrow |\phi|_0 = \sqrt{\frac{-m^2}{2\lambda}} = a$$

The moment you choose a particular vacuum, you talk about fluctuations about vacuum.

Choose $\phi_0 = a$, separating the field into real & imaginary parts,

$$\phi = (l + ia)e^{i\theta} \quad l, \theta \text{ are real fields}$$

$$\partial_\mu \phi \partial^\mu \phi^* = |[\partial_\mu l + (l + ia)i\partial_\mu \theta]e^{i\theta}|^2$$

$$= (\partial_\mu l)^2 + (l + a)^2 (\partial_\mu \theta)^2$$

$$(\partial_\mu l)^2 = \partial_\mu l \partial^\mu l \rightarrow \approx (\partial_\mu l)^2 + a^2 (\partial_\mu \theta)^2$$

Then, $V(\phi) = m^2 (l + a)^2 + \lambda (l + a)^4$
 $= (m^2 a^2 + \lambda a^4) + (2m^2 a^2 + 4\lambda a^3)l + (m^2 + 6\lambda a^2)l^2 + \dots$

$$a^2 = -\frac{m^2}{2\lambda} \Rightarrow \text{The linear term vanishes.}$$

$$V(\Phi) = -\lambda a^4 + 4\lambda a^2 \rho^2 + \dots$$

$$\mathcal{L} \propto (\partial_\mu \rho)^2 + a^2 (\partial_\mu \theta)^2 - 4\lambda a^2 \rho^2 + \dots$$

	ρ	mass ²	$4\lambda a^2$
$\uparrow$	θ	0	0

This is a Lagrangian with 2 parts one for ρ , one for θ .

There is a mass term for ρ .

There is no mass term for θ .

So, when the symmetry is broken, there is no more 2 massive particles $\rightarrow$ one massive and one massless.

$\Rightarrow$ Goldstone Boson (massless)

Let's say the field is, $\Phi = a + \frac{\phi_1 + i\phi_2}{\sqrt{2}}$
 $\uparrow$
fluctuation from vacuum

$$\partial_\mu \Phi = \frac{(\partial_\mu \phi_1) + i\partial_\mu \phi_2}{\sqrt{2}} \Rightarrow \partial_\mu \Phi \partial^\mu \Phi^* = \frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} (\partial_\mu \phi_2)^2$$

$$|\Phi|^2 = \frac{\phi_1^2 + \phi_2^2}{2} + \sqrt{2}a\phi_1 + a^2$$

$$V(\Phi) = m^2 |\Phi|^2 + \lambda |\Phi|^4 = \cancel{(m^2 a^2 + \lambda a^4)}$$

$$= (m^2 a^2 + \lambda a^4) + (\sqrt{2}am^2 + 2\sqrt{2}\lambda a^3) \phi_1 + \left(m^2 \frac{\phi_1^2 + \phi_2^2}{2} + \lambda 2a^2 \phi_1^2 + \lambda a^2 (\phi_1^2 + \phi_2^2) \right)$$

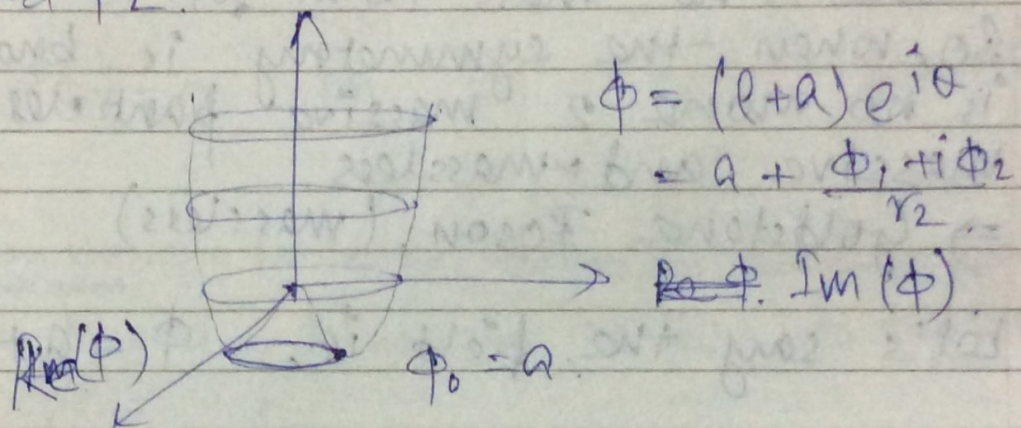
$$\text{Then, } \mathcal{L} = \frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} (\partial_\mu \phi_2)^2 - 2\lambda a^2 \phi_1^2$$

$\phi_1 \rightarrow$ has a mass term, $\phi_2 \rightarrow$ doesn't.

Comparing the two,

$$\ell + a + ia\theta = a + \frac{\phi_1}{\sqrt{2}} + \frac{i\phi_2}{\sqrt{2}}$$

31/3/16 If you change ℓ , then you are moving along the axis, so there is an energy cost. But if you change θ , you are in the same plane, there is no energy cost. Energy is related to the mass. So ℓ has a mass term, θ does not. Similarly, for ϕ_1 and ϕ_2 .



For N scalar fields, $\sum_{i=1}^N (\partial_\mu \phi_i)^2$

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} = \Phi \quad (\partial_\mu \Phi)^T (\partial^\mu \Phi)$$

For a global symmetry, $\Phi \rightarrow O\Phi$ $O \in SO(N)$
 $\partial_\mu \Phi \rightarrow O\partial_\mu \Phi$

$$\mathcal{L} = \sum_{i=1}^N (\partial_\mu \phi_i)^2 - V(\Phi)$$

$$V(\Phi) = m^2 (\Phi^T \Phi) + \lambda (\Phi^T \Phi)^2$$

$\Phi^T \Phi$ is like $|\Phi|^2$

$\Phi \rightarrow O\Phi$ If you want the vacuum state to be invariant under this transform, then, $(\Phi^T)_0 \Phi_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ can be a choice.

For $m^2 < 0$, $(\phi^\dagger \phi)_0 = -\frac{m^2}{2\lambda} = a^2$

We have vacuum which are column vectors whose lengths are $a^2 \rightarrow$ infinite choice.

Let us take $\phi_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a \end{pmatrix} \leftarrow$ This state is not invariant when operated upon by O . But the length is preserved.

$\phi = \phi_0 + \chi \rightarrow$ fluctuation from vacuum

$$\begin{aligned} \phi^\dagger \phi &= \phi_0^\dagger \phi_0 + 2\phi_0^\dagger \chi + \chi^\dagger \chi \\ &= \chi^\dagger \chi + 2a \chi_N + a^2 \end{aligned}$$

$$\Rightarrow V(\phi) = (m^2 a^2 + \lambda a^4) + (2a m^2 \chi_N + \lambda 4a^3 \chi_N) + [m^2 (\chi^\dagger \chi) + \lambda 4a^2 \chi_N^2 + 2\lambda a^2 \chi^\dagger \chi] \therefore$$

$$\Rightarrow V(\phi) \approx 4\lambda a^2 \chi_N^2$$

$$\mathcal{L} = \partial_\mu \chi \partial^\mu \chi - 4\lambda a^2 \chi_N^2 + \dots$$

From this N fields, only one has mass, so $(N-1)$ Goldstone Bosons.

$SO(N)$ $\leftarrow$ Broken

$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a \end{pmatrix} \leftarrow SO(N-1) \leftarrow$ Residual. (Any rotation which does not affect N th comp.)

$$V(\Phi) = V(\Phi_0) + \frac{1}{2!} \left(\frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j} \right)_{\Phi_0} \chi_i \chi_j + \dots$$

$$\chi = \Phi - \Phi_0$$

$$\Rightarrow V(\Phi) = V(\Phi_0) + \frac{1}{2} M_{ij} \chi_i \chi_j$$

$$\Rightarrow \delta V = \frac{1}{2} \chi^T M \chi \leftarrow \text{Always positive (!)}$$

↑ semi-positive definite

$$\chi \text{ has to be } \chi = \sum c_i v_i \Rightarrow \chi^T M \chi = \sum c_i^2 \lambda_i$$

non negative.

$$\text{For } \chi^T M \chi = 0 \Rightarrow M \chi = 0$$

$$\sum c_i^2 \lambda_i = 0 \rightarrow \text{sum of } \lambda_i \text{ have to be } 0 \text{ (}\lambda_i \text{ have to be } 0\text{)}$$

Eigenvalues have to be 0 $\Rightarrow \chi$ eigenvectors of 0 eigenvalue.

$$\Phi = (e^{-ig\theta^a T_a}) \Phi_0 \text{ (changing } \Phi_0 \text{ by group operation)}$$

K.E and P.E should be invariant under this transformation if it is a symmetry.

$$\Rightarrow V(\Phi) = V(\Phi_0)$$

$$\Phi = \Phi_0 - ig\theta^a T_a \Phi_0 \Rightarrow \chi = -ig\theta^a (T_a \Phi_0)$$

$$\text{As } \delta V \neq 0 \Rightarrow \chi^T M \chi = 0 \Rightarrow M(T_a \Phi_0) = 0.$$

Every generation of the group T_a acting on Φ_0 gives a eigenvector. This transformation does not change the vacuum.

If $T_a \Phi_0 = 0$, T_a kills Φ_0 , then you cannot further comment about it.

It is an eigenvector of M unless $\Phi_0 = 0$.

If there are $|G|$ generators in the unbroken symmetry group and H in the isotropic subgroup, then there are at least $(G-H)$.

non-zero eigenvectors + zero vectors. So, you can say that there are at least $|G| - |H|$ no. of Goldstone Bosons (which have n eigenvalue 0).

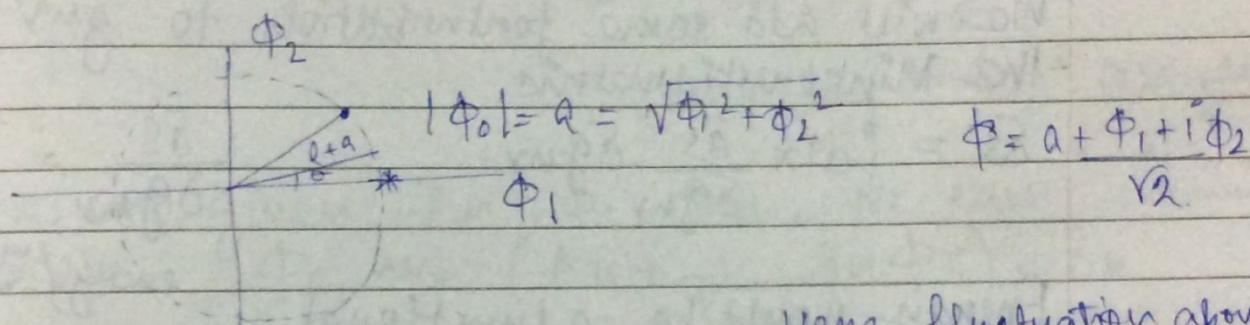
$$\# \text{ Goldstone Bosons} = |G| - |H|$$

$$\mathcal{L} = (\partial_\mu + ieA_\mu) \phi (\partial^\mu - ieA^\mu) \phi^\dagger - V(\phi)$$

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$V(\phi) = m^2 |\phi|^2 + \lambda |\phi|^4$$

$$m^2 < 0, \quad (\phi = a + \frac{\phi_1 + i\phi_2}{\sqrt{2}})$$



$\phi = (r+a)e^{i\theta}$ will also work $\leftarrow$ Here, fluctuation about vacuum $\rightarrow$ $r+a$ length, θ angle

Why not, $\phi = a + re^{i\theta}$?

$\rightarrow$ Here $\rightarrow$ the length is r , θ can be as large as you want. θ can be either multi-valued or discontinuous. It will be close to vacuum value even if θ is large $\rightarrow +\phi^e a$

In 1D quantum mechanics, there is no spontaneous symmetry breaking.

$\rightarrow$ 2 minima are there, you can expand about either of the minima but the final wavefunction is the superposition of both of these.

The ground states are not oblivious of each other due to tunnelling. QFT makes this prob. 0.

U(1) gauge theory with SSB

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi) (D^\mu \phi)^* - V(\phi)$$

where,

$$D_\mu = \partial_\mu + ie A_\mu, \quad V(\phi) = m^2 |\phi|^2 + \lambda |\phi|^4 \quad \begin{matrix} \lambda > 0 \\ m^2 < 0 \end{matrix}$$

$$A_\mu \rightarrow A'_\mu = A_\mu + \frac{1}{e} \partial_\mu \alpha$$

$$\phi \rightarrow \phi' = e^{-i\alpha} \phi$$

$$|\phi| = \sqrt{\frac{m^2}{2\lambda}} = a$$

$$E = \int d^3x \cdot T^{00}$$

$T^{\mu\nu} \rightarrow$ Energy momentum tensor

$$\text{Action, } S = \int d^4x \sqrt{-g} \mathcal{L}$$

We will add some perturbation to g_{μν} from the Minkowski metric

$$\delta S = \int d^4x \frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu}$$

$$\frac{\delta S}{\delta g_{\mu\nu}} = \overset{\substack{\uparrow \\ \text{symmetric} \\ \text{energy-momentum tensor}}}{T_{\mu\nu}}$$

Energy would be a functional,

$$E[A_\mu, \phi] = \int d^3x \left[\frac{1}{2} \sum_i (F_{0i})^2 + \frac{1}{4} \sum_{ij} (F_{ij})^2 + |D_0 \phi|^2 + \sum_i |D_i \phi|^2 + V(\phi) \right]$$

comes from, $\frac{\partial}{\partial g_{00}} (\sqrt{-g} \mathcal{L})$

In electromagnetism, $F_{0i} \rightarrow$ electric fields
 $F_{ij} \rightarrow$ magnetic field comp

For this to be minimum, $F_{\mu\nu} = 0$

$$\Rightarrow A_\mu = \frac{1}{e} \partial_\mu \alpha \quad \text{Pure Gauge}$$

$D_0 \phi, D_i \phi$ have to be 0

$$D_\mu \phi = (\partial_\mu + ie \frac{1}{e} \partial_\mu \alpha) \phi = 0$$

$$\boxed{\phi = e^{-i\alpha(x)} \phi_0}$$

And, minimising $V(\phi)$, $|\phi_0| = \sqrt{\frac{-m^2}{2\lambda}}$

Let's choose, $A^\mu_\mu = 0$ and $\phi^\mu = a$ (Vacuum)

$$\phi = a + \frac{\phi_1 + i\phi_2}{\sqrt{2}}$$

$$D_\mu \phi \approx \frac{1}{\sqrt{2}} (\partial_\mu \phi_1 + i\partial_\mu \phi_2) + i\sqrt{2}eaA_\mu$$

Only upto second order of field in $\mathcal{L}$

We want upto quadratic terms of field in the Lagrangian. Then $D_\mu \phi$ should have linear term of field.

$$\text{Then, } \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} 2e^2 a^2 \left(A_\mu + \frac{1}{\sqrt{2}ea} \partial_\mu \phi_2 \right)^2 - (2\lambda a^2 - m^2) \phi_1^2$$

As in last discussion

If we are given, A_μ, ϕ_2 , then, we can calculate ϕ_1 and $\left(A_\mu + \frac{1}{\sqrt{2}ea} \partial_\mu \phi_2 \right) = B_\mu$

$$\partial_\mu B_\nu - \partial_\nu B_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} = B_{\mu\nu}$$

$$\text{Then, } \mathcal{L} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} (\partial_\mu \phi_1)^2 + e^2 a^2 B_\mu B^\mu - (2\lambda a^2 - m^2) \phi_1^2$$

The field ϕ_2 has disappeared from the Lagrangian, there is no constraint on it anymore

Higg's mechanism $\rightarrow$ The mass term for ϕ_2 can not be there because of gauge symmetry. On the other hand $B_{\mu\nu}$ has mass term

Now B_μ has a mass term $\rightarrow$ Higg's potential (?)

A_μ, ϕ_1, ϕ_2 } Equivalent descriptions of the system
 B_μ, ϕ_1, ϕ_2 }

$$\mathcal{L}^{(2)} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + e^2 a^2 B_\mu B^\mu + \frac{1}{2} (\partial_\mu \Phi_1)^2 - 2\lambda a^2 \Phi_1^2$$

Only one massive scalar Boson, no Goldstone Boson.

$$(*) \quad \Phi \rightarrow \Phi' = \Phi - i\alpha\Phi$$

$$A_\mu \rightarrow A_\mu' = A_\mu + \frac{1}{e} \partial_\mu \alpha$$

After the transformation, $\Phi' = a + \frac{\Phi_1' + \Phi_2'}{\sqrt{2}}$

$$= a + \frac{\Phi_1 + i\Phi_2}{\sqrt{2}} - i\alpha a \quad (\text{small fluctuation ignoring quadratic order})$$

$$\Phi_1' = \Phi_1$$

$$\Phi_2' = \Phi_2 - \sqrt{2}\alpha a$$

$$B_\mu' = A_\mu' + \frac{1}{\sqrt{2}ea} \partial_\mu \Phi_2'$$

$$= A_\mu + \frac{1}{e} \partial_\mu \alpha + \frac{1}{\sqrt{2}ea} (\partial_\mu \Phi_2 - \sqrt{2}a \partial_\mu \alpha)$$

$$= B_\mu$$

In the Lagrangian, we have fields (B_μ and Φ_1) which do not change under transformation $B_\mu \rightarrow$ gauge scalar.

But, this Lagrangian is only upto quadratic order correct only upto lower order, not upto full order.

$$\Phi(x) = \rho(x) e^{i\theta(x)}$$

So, we use polar decomposition only

when Φ is nonzero throughout.

If $\Phi = 0$, then for finding fluctuation, ρ might be very small, but θ might be very diff.

Full Lagrangian without any approx.

7/4/16

For vacuum, $\phi^{(0)} = a$
 $\rho^{(0)} = a$
 $\theta^{(0)} = 0$

$$A_\mu' = A_\mu + \frac{1}{e} \partial_\mu \alpha$$

$$\phi' = e^{-i\alpha} \phi$$

$$\rho' e^{i\theta'} = \rho e^{i(\theta - \alpha)} \Rightarrow \rho' = \rho, \theta' = \theta - \alpha$$

$$D_\mu \phi = \partial_\mu \phi + ie A_\mu \phi$$

$$= (\partial_\mu \rho + i\rho \partial_\mu \theta + ie A_\mu \rho) e^{i\theta}$$

$$= [\partial_\mu \rho + i\rho (\partial_\mu \theta + e A_\mu)] e^{i\theta}$$

full
Lagrangian
without
any approx

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\partial_\mu \phi)(\partial^\mu \rho) + \rho^2 e^2 (A_\mu + \frac{1}{e} \partial_\mu \theta)^2 - V(\rho)$$

let's define $\tilde{B}_\mu = A_\mu + \frac{1}{e} \partial_\mu \theta$, $F_{\mu\nu} = \tilde{B}_{\mu\nu}$

$$\text{Then, } \mathcal{L} = -\frac{1}{4} \tilde{B}_{\mu\nu} \tilde{B}^{\mu\nu} + \rho^2 e^2 \tilde{B}_\mu \tilde{B}^\mu + \partial_\mu \rho \partial^\mu \rho - V(\rho)$$

7/4/16 $SU(2)$ global symmetry $\rightarrow$

$$\mathcal{L} = (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) - V(\Phi)$$

$$= |\partial_\mu \Phi_1|^2 + |\partial_\mu \Phi_2|^2 - V(\Phi)$$

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$\partial_\mu \Phi = \begin{pmatrix} \partial_\mu \phi_1 \\ \partial_\mu \phi_2 \end{pmatrix}$$

$$V(\Phi) = m(\Phi^\dagger \Phi) + \lambda(\Phi^\dagger \Phi)^2$$

Symmetry under $\Phi \rightarrow \omega \Phi$: $\omega \in SU(2)$

$$\omega^\dagger \omega = \mathbb{I}$$

$$(\lambda > 0)$$

$$\partial_\mu \Phi \rightarrow \omega \partial_\mu \Phi$$

SU(2) local (gauge) symmetry

$$\mathcal{L} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi) - \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a}$$

$$D_\mu = \partial_\mu + ig A_\mu \equiv \left(\partial_\mu + ig \frac{\tau_a}{2} A_\mu^a \right) \Phi$$

$$\Phi \rightarrow \Phi' = w(x) \Phi$$

$$(D_\mu \Phi) \rightarrow (D_\mu \Phi)' = w(x) (D_\mu \Phi)$$

$$A_\mu' = w A_\mu w^{-1} + \frac{i}{g} (\partial_\mu w) w^{-1} \\ = w A_\mu w^{-1} - \frac{i}{g} w (\partial_\mu w^{-1})$$

Expand in the basis of generators

$$A_\mu = T_a A_\mu^a \leftarrow \text{ordinary fields (function)}$$

$$F_{\mu\nu} = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) T_a + ig [T_b, T_c] A_\mu^b A_\nu^c$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig [A_\mu, A_\nu]$$

Extra factor in non-abelian theory

$$F_{\mu\nu} = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f_{bc}^a A_\mu^b A_\nu^c) T_a$$

$$F_{\mu\nu}^a$$

$$T_a F_{\mu\nu} F^{\mu\nu} = F_{\mu\nu}^a F^{\mu\nu b} T_a (T_a T_b) \quad \left| \begin{array}{l} T_a (T_a T_b) \\ = \frac{\delta_{ab}}{2} \end{array} \right.$$

We need to minimize energy

$$\mathcal{E}[A_\mu^a, \Phi] = \int d^3x \left[\frac{1}{2} \sum_{i,a} (\partial_0 A_i^a)^2 + \frac{1}{4} \sum_{i,j,a} (F_{ij}^a)^2 + (D_0 \Phi)^\dagger (D_0 \Phi) + \sum_i (D_i \Phi)^\dagger (D_i \Phi) + V(\Phi) \right]$$

$$\Rightarrow F_{\mu\nu}^a = 0 \quad (\text{In vacuum, since the term except the last term needs to be 0})$$

This implies A_μ has to be a pure gauge (not $A_\mu = 0$)

$$A_\mu \rightarrow A_\mu' = w A_\mu w^{-1} - \frac{i}{g} w (\partial_\mu w^{-1}) \quad \left| A_\mu = -\frac{i}{g} w (\partial_\mu w^{-1}) \right.$$

$$D_\mu \Phi = 0$$

For this particular A_μ ,

$$D_\mu \phi = \partial_\mu \phi + i g (\partial_\mu w^{-1}) \phi = 0$$

$$w^{-1} \partial_\mu \phi + (\partial_\mu w^{-1}) \phi = 0$$

$$\Rightarrow \partial_\mu (w^{-1} \phi) = 0 \Rightarrow \phi = w(x) \phi_0 \leftarrow \text{constant column matrix}$$

$$\phi_0^\dagger \phi_0 = \frac{-m^2}{2\lambda} = a^2$$

Vacuum: $A_\mu = -\frac{i}{g} w (\partial_\mu w^{-1})$

$$\phi = w \phi_0$$

$$\phi_0^\dagger \phi_0 = a^2$$

choice: $A_\mu^{(0)} = 0$, $\phi^{(0)} = \begin{pmatrix} 0 \\ a \end{pmatrix}$. The moment you choose one, symmetry (global) is broken

Fluctuation about the vacuum,

$$\phi(x) = \begin{pmatrix} \xi_1 + i \xi_2 \\ a + \xi_3 + i \xi_4 \end{pmatrix} \quad \text{where } \xi_1, \xi_2, \xi_3, \xi_4 \rightarrow \text{small}$$

$$\phi(x) = \underset{\substack{\uparrow \\ \text{SU}(2) \\ \text{matrix close to } \mathbb{I}}}{w(x)} \begin{pmatrix} 0 \\ a + \chi \end{pmatrix} \quad (\chi \rightarrow \text{Real field})$$

Thing

$\begin{pmatrix} 0 \\ a + \chi \end{pmatrix}$ unitary gauge. (By choosing a particular gauge choice)

Change in A_μ^a 's going to be small $\Rightarrow A_\mu^a$'s are small

$$w(x) = \left(\mathbb{I} + i g \frac{T_a}{2} u^a \right)$$

$$\Rightarrow w(x) \begin{pmatrix} 0 \\ a + \chi \end{pmatrix} = \begin{pmatrix} 0 \\ a + \chi \end{pmatrix} + \frac{i g}{2} \begin{pmatrix} u^3 & u^1 - i u^2 \\ u^1 + i u^2 & -u^3 \end{pmatrix} \begin{pmatrix} 0 \\ a + \chi \end{pmatrix}$$

$$= \begin{pmatrix} \frac{i g a \chi}{2} + \frac{g a u^2}{2} \\ a + \chi - \frac{i g a u^3}{2} \end{pmatrix}$$

you can compare this to $\xi_1, \xi_2, \xi_3, \xi_4$

We want to find out $\mathcal{L}^{(2)}$.

$$D_\mu \Phi = \begin{pmatrix} 0 \\ \partial_\mu \chi \end{pmatrix} + \begin{pmatrix} i g A_\mu^1 + g \frac{A_\mu^2}{2} \\ -i g \frac{A_\mu^3}{2} \end{pmatrix} \Phi$$

$$D_\mu = \partial_\mu + i g \frac{T_a}{2} A_\mu^a$$

$$V(\Phi) = 4 \lambda a^2 \chi^2$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a$$

$$= f_{\mu\nu}^a \quad (A_\mu A_\nu^a \text{ and } A_\nu A_\mu^a \text{ are small})$$

The commutator gives higher order.

Degrees of freedom $\left. \begin{matrix} \Phi_1 \\ \Phi_2 \end{matrix} \right\} \rightarrow 4$ Degrees of freedom

A_μ 's $\rightarrow 3$ generators (each has 2 degrees).
massless spin $\frac{1}{2} \rightarrow 6$.

Total Degrees of freedom $\rightarrow 6 + 4 = 10$.

11/4/16 $A_\mu^{(a)} \rightarrow \text{small} \quad \begin{pmatrix} 0 \\ a + \chi(n) \end{pmatrix} = \Phi$

$$V(\Phi) = 4 \lambda a^2 \chi^2 \quad (\text{In quadratic terms})$$

The linear order term of $F_{\mu\nu}^a$ is,

$$(F_{\mu\nu}^a)^{(1)} = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g \epsilon^{abc} A_\mu^b A_\nu^c)^{(1)}$$

$$= f_{\mu\nu}^a$$

$$\mathcal{L}^{(2)} = -\frac{1}{4} f_{\mu\nu}^a f^{\mu\nu a} + (D_\mu \Phi)^{(1)} (D^\mu \Phi)^{(1)} + V(\Phi)$$

$$(D_\mu \Phi)^{(1)} = \begin{pmatrix} 0 \\ \partial_\mu \chi \end{pmatrix} + \frac{i g}{2} \begin{pmatrix} A_\mu^3 & A_\mu^1 - i A_\mu^2 \\ A_\mu^1 + i A_\mu^2 & -A_\mu^3 \end{pmatrix} \begin{pmatrix} 0 \\ a \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{i g a}{2} (A_\mu^1 - i A_\mu^2) \\ \partial_\mu \chi - \frac{i g a}{2} A_\mu^3 \end{pmatrix} \quad (\text{Ignoring higher order terms like } A_\mu \chi)$$

$$\mathcal{L}^{(2)} = -\frac{1}{4} f_{\mu\nu}^a f^{\mu\nu a} + (\partial_\mu \chi)^2 - 4 \lambda a^2 \chi^2 + \frac{g^2 a^2}{4} (A_\mu^a)^2$$

The χ is clearly an equation for a massive particle.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \alpha A_\mu A^\mu$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} &= \frac{\partial}{\partial(\partial_\mu A_\nu)} \left(-\frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} \right) = -\frac{1}{2} F^{\rho\sigma} \frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)} \\ &= -\frac{1}{2} (\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho) F^{\rho\sigma} \quad \left(\frac{\partial(\partial_\rho A_\sigma)}{\partial(\partial_\mu A_\nu)} \right) \\ &= -\frac{1}{2} (F^{\mu\nu} - F^{\nu\mu}) = -F^{\mu\nu} \end{aligned}$$

$$-\partial_\mu (F^{\mu\nu}) = 2\alpha A^\nu$$

$$\Rightarrow -\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = 2\alpha A^\nu$$

$$(\square + 2\alpha) A^\nu - \partial^\nu (\partial_\mu A^\mu) = 0$$

3 indep fields instead of 4 due to the constraint
If $\partial_\mu A^\mu = 0$, then you end up with
 $(\square + 2\alpha) A^\nu = 0$, similar to $(\square + m^2)\phi = 0$ eq. for massive particle

Then, the mass term, $m^2 = 2\alpha$.

$$\text{Here, } M_{A_\mu}^2 = \frac{g^2 a^2}{4} \Rightarrow M_{A_\mu} = \frac{ga}{2}$$

$U(x) \begin{pmatrix} 0 \\ a \end{pmatrix}$ Other than $\mathbb{I}$, is there any other $SU(2)$ matrix which keeps $\begin{pmatrix} 0 \\ a \end{pmatrix}$ as the same.

$$\begin{aligned} U(x) \begin{pmatrix} 0 \\ a \end{pmatrix} &\simeq \left(\mathbb{I}_2 + i g \frac{T_a}{2} u^a \right) \begin{pmatrix} 0 \\ a \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ a \end{pmatrix} + \frac{ig}{2} \begin{pmatrix} u^3 & u^1 - iu^2 \\ u^1 + iu^2 & -u^3 \end{pmatrix} \begin{pmatrix} 0 \\ a \end{pmatrix} \end{aligned}$$

If $a \neq 0$, then $u_1, u_2, u_3 = 0$.
 Apart from $\mathbb{I}$, all $su(2)$ matrices shift vacuum $\rightarrow$ global symmetry is broken.

$$= \begin{pmatrix} 0 \\ a \end{pmatrix} + \frac{ig}{2} \begin{pmatrix} u^1 a - i u^2 a \\ -u^3 a \end{pmatrix}$$

$SU(2) \times U(1) \rightarrow$ Bosonic sectors:

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad \text{and} \quad V(\Phi) = \lambda (\Phi^\dagger \Phi - a^2)^2$$

just a constant

Without the gauge interaction $(\partial_\mu \Phi)^\dagger (\partial^\mu \Phi)$

$\Phi \rightarrow e^{i\alpha} \Phi$ [$U(1)$ gauge symmetry]

$SU(2)$ gauge symmetry in $F_{\mu\nu}$

Locally, both $SU(2)$ and $U(1)$ symmetries can be preserved.

$$D_\mu \Phi = \partial_\mu \Phi + ig \frac{\tau_a}{2} A_\mu^a \Phi + \frac{ig'}{2} B_\mu \Phi$$

Earlier we had 3 gauge fields (3 K.E terms)

Now we have 4 gauge fields (4 K.E terms)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_\mu \Phi)^\dagger (D^\mu \Phi) - \lambda (\Phi^\dagger \Phi - a^2)^2$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g \epsilon_{abc} A_\mu^b A_\nu^c$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

For vacuum, $F_{\mu\nu}^a = B_{\mu\nu} = 0$ $D_\mu \Phi = 0$

Choose, $A_\mu^a(w) = 0$ $B_\mu(w) = 0$ $\Phi(w) = \begin{pmatrix} 0 \\ a \end{pmatrix}$

Around the vacuum, $\Phi = \begin{pmatrix} 0 \\ a + \chi \end{pmatrix}$ (unitary gauge)

Here, is it possible to keep $\begin{pmatrix} 0 \\ a \end{pmatrix}$ const. unchanged?

$SU(2) \times U(1)$ element close to $\mathbb{I}$

$$\Rightarrow (\mathbb{I} - \frac{i\tau_a}{2} \cdot u^a - \frac{i}{2} \beta \mathbb{I}) \begin{pmatrix} 0 \\ a \end{pmatrix}$$

$$\begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix} \begin{pmatrix} 0 \\ a \end{pmatrix} \Rightarrow \text{kills } \begin{pmatrix} 0 \\ a \end{pmatrix} \text{ (annihilation matrix)}$$

If we want this matrix to be Hermitian, then $\beta = 0$, $\equiv \alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{\tau^3 + \mathbb{I}}{2}$

So, vacuum is not invariant under all $SU(2) \times U(1)$ matrices but a subset of that generated by $\frac{\tau^3 + \mathbb{I}}{2}$. (Symmetry is not fully broken)

$$SU(2) \Rightarrow \exp(\tau^3) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-i\theta} a \end{pmatrix}$$

The corresponding phase change can be reversed by taking an opposite phase factor i.e. just $U(1)$.

SSB of $SU(2) \times U(1) \rightarrow U(1)$ [Bosonic sector of SM]

3/6/15 Choose vacuum,

$$A_M^{(a)} = 0, \quad B_M = 0 \quad \text{and} \quad \phi = \begin{pmatrix} 0 \\ a \end{pmatrix}$$

Fluctuation about the vacuum,

$$\phi = \begin{pmatrix} 0 \\ a+x \end{pmatrix} \quad A_M^a, B_M \rightarrow \text{small unitary gauge.}$$

↑
small real field.

$$\begin{aligned}
 (D_M \Phi) &:= \partial_M \begin{pmatrix} 0 \\ a+x \end{pmatrix} + \left[\frac{ig}{2} \begin{pmatrix} A_M^3 & A_M - iA_M^2 \\ A_M^1 + iA_M^2 & -A_M^3 \end{pmatrix} \right. \\
 &\quad \left. + \frac{ig'}{2} \begin{pmatrix} B_M & 0 \\ 0 & B_M \end{pmatrix} \right] \begin{pmatrix} 0 \\ a+x \end{pmatrix} \\
 &= \begin{pmatrix} \frac{ig}{2} (A_M^1 - iA_M^2) (a+x) \\ \partial_M x + \frac{i}{2} (g'B_M - gA_M^3) (a+x) \end{pmatrix}
 \end{aligned}$$

We can define a new field,

$$W_M^\pm = \frac{A_M^1 \mp iA_M^2}{\sqrt{2}}$$

where, $(W_M^-) = (W_M^+)^*$ (classical)
For Quantum $\rightarrow$ Hermitian conjugate

Another field, $\frac{gA_M^3 - g'B_M}{\sqrt{g^2 + g'^2}} = Z_M$

And, $\frac{g'A_M^3 + gB_M}{\sqrt{g^2 + g'^2}} = A_M \leftarrow \begin{pmatrix} \text{once the } SU(2) \times U(1) \text{ symmetry is broken this goes into EM theory} \end{pmatrix}$

Let's look at the linear part first,

$$(D_M \Phi)^{(1)} = \begin{pmatrix} \frac{iga}{\sqrt{2}} W_M^+ \\ \partial_M x + \frac{iga}{2} \sqrt{g^2 + g'^2} Z_M \end{pmatrix}$$

$$\text{Then, } (D_\mu \phi)^\dagger (D^\mu \phi) = (\partial_\mu X)^2 + \frac{a^2}{4} (g^2 + g'^2) Z_\mu Z^\mu + \frac{g^2 a^2}{2} W_\mu^+ W_\mu^-$$

The last 2 terms look like the mass terms

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g \epsilon_{abc} A_\mu^b A_\nu^c$$

$$\left\{ \left[-\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} \right]^{(2)} \right\} =$$

$$W_{\mu\nu}^\pm \simeq W_{\mu\nu}^\pm = \partial_\mu W_\nu^\pm - \partial_\nu W_\mu^\pm$$

$$= \frac{F_{\mu\nu}^1 \mp i F_{\mu\nu}^2}{\sqrt{2}}$$

$$\text{Then, } (F_{\mu\nu}^1)^2 + (F_{\mu\nu}^2)^2 = 2 W_{\mu\nu}^+ W_{\mu\nu}^-$$

$$(F_{\mu\nu}^3)^2 + (B_{\mu\nu})^2 = (Z_{\mu\nu})^2 + (F_{\mu\nu})^2$$

$$Z_{\mu\nu} = \frac{g F_{\mu\nu}^3 - g' B_{\mu\nu}}{\sqrt{g^2 + g'^2}}$$

$$Z_\mu = \cos \theta_w A_\mu^3 - \sin \theta_w B_\mu$$

$$A_\mu = \sin \theta_w A_\mu^3 + \cos \theta_w B_\mu$$

$$\text{Then, } \left(-\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} \right)^{(2)} = -\frac{1}{2} W_{\mu\nu}^+ W_{\mu\nu}^- - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\begin{aligned} \mathcal{L}^{(2)} = & -\frac{1}{2} W_{\mu\nu}^+ W_{\mu\nu}^- + \frac{g^2 a^2}{2} W_{\mu\nu}^+ W_{\mu\nu}^- \\ & -\frac{1}{4} Z_{\mu\nu} Z_{\mu\nu} + \frac{a^2}{4} (g^2 + g'^2) Z_{\mu\nu} Z_{\mu\nu} \\ & -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} \\ & + (\partial_\mu \chi)^2 - \frac{1}{2} \lambda a^2 \chi^2 \end{aligned}$$

At first you had, $F_{\mu\nu}$, $B_{\mu\nu}$ and 2 scalar fields Φ, Φ^+ .

After this, we have, $W_{\mu\nu}^+, W_{\mu\nu}^-$ which have mass, $Z_{\mu\nu}$ has mass. (3 gauge fields)

The field A_μ (from $F_{\mu\nu}$) don't have mass.

And a field χ (scalar) which is a massive Boson.

The degrees of freedom have not changed.

Before symmetry breaking,

2 massive complex fields $\Phi_1, \Phi_2 \rightarrow 4$ DOF.

4 gauge fields, each massless $\rightarrow 4 \times 2$ DOF.

Total DOF = 12.

(?) 3 Goldstone Bosons get eaten up 3 gauge Bosons (acquire mass) $\rightarrow$ so DOF remain same after symmetry breaking.

$$m_W = \frac{ga}{\sqrt{2}}$$

$$m_Z = \frac{\sqrt{g^2 + g'^2} a}{\sqrt{2}} \Rightarrow \frac{m_W}{m_Z} = \frac{g}{\sqrt{g^2 + g'^2}}$$

$$m_X = 2a\sqrt{\lambda}$$

We had,

$$Z_\mu = \cos \theta_W A_\mu^3 - \sin \theta_W B_\mu$$

$$A_\mu^0 = \sin \theta_W A_\mu^3 + \cos \theta_W B_\mu$$

$$\frac{m_W}{m_Z} = \cos \theta_W$$

$$\sin^2 \theta_W = 0.23$$

$$m_W = 80 \text{ GeV}$$

$$m_Z = 91 \text{ GeV}$$

W Bosons $\rightarrow$ change particle interaction

Z Bosons $\rightarrow$ No change (charge stays the same)

Electron can emit a virtual photon, while scattering with another electron. It violates the energy-momentum conservation. So, the virtual photon can stay for a very short time, so that it doesn't violate. But photon's energy can be made arbitrarily small (as it is a massless particle), so the interaction can be made long-ranged. Again, there is a limit to this arbitrarily small.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_\mu \phi)^\dagger (D^\mu \phi) - \lambda(\phi^\dagger \phi - a^2)$$

The last two terms don't have any A_μ term. $B_{\mu\nu}$ does have (mixed out beyond quadratic order).

$$F_{\mu\nu}^1 = \partial_\mu A_\nu^1 - \partial_\nu A_\mu^1 - g \epsilon_{1ab} A_\mu^a A_\nu^b$$

$$= \partial_\mu A_\nu^1 + g A_\mu^3 A_\nu^2 - (\mu \leftrightarrow \nu)$$

$$F_{\mu\nu}^2 = \partial_\mu A_\nu^2 - \partial_\nu A_\mu^2 - g A_\mu^3 A_\nu^1 - (\mu \leftrightarrow \nu)$$

We had, $W_{\mu\nu}^\pm = \frac{A_\mu^1 \mp i A_\mu^2}{\sqrt{2}}$

$$Z_\mu = \cos \theta_W A_\mu^3 - \sin \theta_W B_\mu$$

$$A_\mu = \sin \theta_W A_\mu^3 + \cos \theta_W B_\mu$$

$$A_\mu^3 = \cos \theta_W Z_\mu + \sin \theta_W A_\mu$$

$$W_{\mu\nu}^\pm = \frac{F_{\mu\nu}^1 \mp i F_{\mu\nu}^2}{\sqrt{2}}$$

$$W_{\mu\nu}^\pm = \partial_\mu W_\nu^\pm - \partial_\nu W_\mu^\pm$$

$$W_{\mu\nu}^\pm = \frac{\partial_\mu A_\nu^1 \mp i \partial_\mu A_\nu^2}{\sqrt{2}} + g A_\mu^3 (A_\nu^2 \pm i A_\nu^1) - (\mu \leftrightarrow \nu)$$

$$= \partial_\mu W_\nu^\pm \pm i g A_\mu^3 W_\nu^\pm - (\mu \leftrightarrow \nu)$$

$$= \partial_\mu W_\nu^\pm \pm i g \sin \theta_W A_\mu^3 W_\nu^\pm \pm i g \cos \theta_W Z_\mu W_\nu^\pm - (\mu \leftrightarrow \nu)$$

It looks like a covariant derivative

$$D_\mu W_\nu^\pm = \partial_\mu W_\nu^\pm \pm ig \sin \theta_W A_\mu W_\nu^\pm$$

$$W'^\pm = e^{\mp i\alpha} W^\pm$$

→ opposite change for $W^\pm$

$$A_\mu' = A_\mu + \frac{1}{g \sin \theta_W} \partial_\mu \alpha$$

$$Z_\mu' \rightarrow Z_\mu$$

no ↑ change. ($\alpha = 0$)

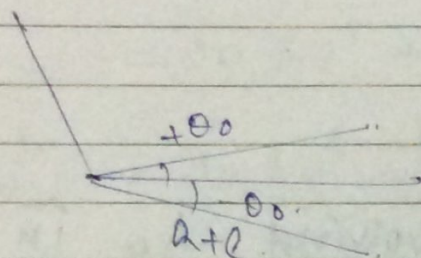
- (*) The other term involving A_μ in $\mathcal{L}$ → show that they are also invariant under these transformations.

$$\mathcal{L}^{(2)} = \partial_\mu \phi \partial^\mu \phi^* - 4a^2 \lambda \phi^2$$

→ mass term → $4a^2 \lambda$

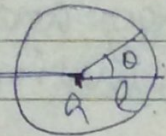
$$\mathcal{L}^{(2)} = \frac{1}{2} (\partial_\mu \phi)^2 - 2\lambda a^2 \phi^2$$

mass term → $2\lambda a^2$ (overall factor. $\frac{1}{2}$ - there)
find out E-L eq. of motion to find out exact mass-term.



Hence, the field stays close to vacuum for small

$$\phi = (a + \epsilon) e^{i\theta}$$

for $(a e^{i\theta} + \epsilon)$ —  θ can be multivalued, still it will be close to vacuum.

Higgs Mechanism → The field ϕ_2 completely disappears from the Lagrangian
So, → no goldstone mode.

$$\mathcal{V}^{(2)} = m_{11} \phi_1^2 + m_{12} \phi_1 \phi_2 + \dots + m_{1n} \phi_1 \phi_n \\ m_{21} \phi_2 \phi_1 + m_{22} \phi_2^2 + \dots + m_{2n} \phi_2 \phi_n$$

$$(\phi_1 \dots \phi_n) \begin{pmatrix} m_{11} & \dots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \dots & m_{nn} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix}$$

M can be orthogonally diagonalised

$$M = O^T M_D O$$

$$\begin{aligned} \Phi^T M \Phi &= \Phi^T O^T M_D O \Phi \\ &= \Phi'^T M_D \Phi' \end{aligned}$$

Orthogonal diagonalisation as $(\partial_\mu \Phi')^T (\partial^\mu \Phi) \rightarrow \Phi'^T M_D \Phi$ stays invariant.

$$\Rightarrow \chi^T M \chi = 0.$$

why $M\chi = 0$?

$$\chi = c_1 \Phi_1 + c_2 \Phi_2 + \dots + c_n \Phi_n$$

$$\chi^T M \chi = 0 = c_1^2 m_1 + c_2^2 m_2 + \dots + c_n^2 m_n$$

$m_1, m_2, \dots$ are guaranteed to be 0. otherwise energy will be negative.

(*) $\partial_\mu F^{\mu\nu} - m^2 A^\nu = 0$. (for massive particles)

$$\partial_\nu \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) - m^2 \partial_\nu A^\nu = 0$$